International Journal of Physical Sciences

Volume 10 Number 24 30 December 2015

ISSN 1992-1950

Academic Journals

ABOUT IJPS

The **International Journal of Physical Sciences (IJPS)** is published weekly (one volume per year) by Academic Journals.

International Journal of Physical Sciences (IJPS) is an open access journal that publishes high-quality solicited and unsolicited articles, in English, in all Physics and chemistry including artificial intelligence, neural processing, nuclear and particle physics, geophysics, physics in medicine and biology, plasma physics, semiconductor science and technology, wireless and optical communications, materials science, energy and fuels, environmental science and technology, combinatorial chemistry, natural products, molecular therapeutics, geochemistry, cement and concrete research, metallurgy, crystallography and computer-aided materials design. All articles published in IJPS are peer-reviewed.

Contact Us

Editorial Office: ijps@academicjournals.org

Help Desk: helpdesk@academicjournals.org

Website: http://www.academicjournals.org/journal/IJPS

Submit manuscript online http://ms.academicjournals.me/

Editors

Prof. Sanjay Misra
Department of Computer Engineering, School of Information and Communication Technology Federal University of Technology, Minna, Nigeria.

Prof. Songjun Li
School of Materials Science and Engineering, Jiangsu University, Zhenjiang, China

Dr. G. Suresh Kumar
Senior Scientist and Head Biophysical Chemistry Division Indian Institute of Chemical Biology (IICB)(CSIR, Govt. of India), Kolkata 700 032, INDIA.

Dr. 'Remi Adewumi Oluyinka
Senior Lecturer, School of Computer Science Westville Campus University of KwaZulu-Natal Private Bag X54001 Durban 4000 South Africa.

Prof. Hyo Choi
Graduate School Gangneung-Wonju National University Gangneung, Gangwondo 210-702, Korea

Prof. Kui Yu Zhang
Laboratoire de Microscopies et d'Etude de Nanostructures (LMEN) Département de Physique, Université de Reims, B.P. 1039. 51687, Reims cedex, France.

Prof. R. Vittal
Research Professor, Department of Chemistry and Molecular Engineering Korea University, Seoul 136-701, Korea.

Prof Mohamed Bououdina
Director of the Nanotechnology Centre University of Bahrain PO Box 32038, Kingdom of Bahrain

Prof. Geoffrey Mitchell
School of Mathematics, Meteorology and Physics Centre for Advanced Microscopy University of Reading Whiteknights, Reading RG6 6AF United Kingdom.

Prof. Xiao-Li Yang
School of Civil Engineering, Central South University, Hunan 410075, China

Dr. Sushil Kumar
Geophysics Group, Wadia Institute of Himalayan Geology, P.B. No. 74 Dehra Dun - 248001(UC) India.

Prof. Suleyman KORKUT
Duzce University Faculty of Forestry Department of Forest Industrial Engineeering Beciyorukler Campus 81620 Duzce-Turkey

Prof. Nazmul Islam
Department of Basic Sciences & Humanities/Chemistry, Techno Global-Balurghat, Mangalpur, Near District Jail P.O: Beltalapark, P.S: Balurghat, Dist.: South Dinajpur, Pin: 733103,India.

Prof. Dr. Ismail Musirin
Centre for Electrical Power Engineering Studies (CEPES), Faculty of Electrical Engineering, Universiti Teknologi Mara, 40450 Shah Alam, Selangor, Malaysia

Prof. Mohamed A. Amr
Nuclear Physic Department, Atomic Energy Authority Cairo 13759, Egypt.

Dr. Armin Shams
Artificial Intelligence Group, Computer Science Department, The University of Manchester.

Editorial Board

Prof. Salah M. El-Sayed
Mathematics. Department of Scientific Computing.
Faculty of Computers and Informatics,
Benha University. Benha ,
Egypt.

Dr. Rowdra Ghatak
Associate Professor
Electronics and Communication Engineering Dept.,
National Institute of Technology Durgapur
Durgapur West Bengal

Prof. Fong-Gong Wu
College of Planning and Design, National Cheng Kung University
Taiwan

Dr. Abha Mishra.
Senior Research Specialist & Affiliated Faculty.
Thailand

Dr. Madad Khan
Head
Department of Mathematics
COMSATS University of Science and Technology
Abbottabad, Pakistan

Prof. Yuan-Shyi Peter Chiu
Department of Industrial Engineering & Management
Chaoyang University of Technology
Taichung, Taiwan

Dr. M. R. Pahlavani,
Head, Department of Nuclear physics,
Mazandaran University,
Babolsar-Iran

Dr. Subir Das,
Department of Applied Mathematics,
Institute of Technology, Banaras Hindu University,
Varanasi

Dr. Anna Oleksy
Department of Chemistry
University of Gothenburg
Gothenburg,
Sweden

Prof. Gin-Rong Liu,
Center for Space and Remote Sensing Research
National Central University, Chung-Li,
Taiwan 32001

Prof. Mohammed H. T. Qari
Department of Structural geology and remote sensing
Faculty of Earth Sciences
King Abdulaziz UniversityJeddah,
Saudi Arabia

Dr. Jyhwen Wang,
Department of Engineering Technology and Industrial Distribution
Department of Mechanical Engineering
Texas A&M University
College Station,

Prof. N. V. Sastry
Department of Chemistry
Sardar Patel University
Vallabh Vidyanagar
Gujarat, India

Dr. Edilson Ferneda
Graduate Program on Knowledge Management and IT,
Catholic University of Brasilia,
Brazil

Dr. F. H. Chang
Department of Leisure, Recreation and Tourism Management,
Tzu Hui Institute of Technology, Pingtung 926,
Taiwan (R.O.C.)

Prof. Annapurna P.Patil,
Department of Computer Science and Engineering,
M.S. Ramaiah Institute of Technology, Bangalore-54,
India.

Dr. Ricardo Martinho
Department of Informatics Engineering, School of Technology and Management, Polytechnic Institute of Leiria, Rua General Norton de Matos, Apartado 4133, 2411-901 Leiria,
Portugal.

Dr Driss Miloud
University of mascara / Algeria
Laboratory of Sciences and Technology of Water
Faculty of Sciences and the Technology
Department of Science and Technology
Algeria

Prof. Bidyut Saha,
Chemistry Department, Burdwan University, WB,
India

International Journal of Physical Sciences

Table of Contents: Volume 10 Number 24, 30 December, 2015

ARTICLES

academicJournals

Vol. 10(24), pp. 590-603, 30 December, 2015
DOI: 10.5897/IJPS2015.4298
Article Number: E0B03B556725
ISSN 1992 - 1950
Copyright ©2015
Author(s) retain the copyright of this article
http://www.academicjournals.org/IJPS

International Journal of Physical Sciences

Full Length Research Paper

Analytical and numerical study of a pulsatile flow in presence of a magnetic field

Mohamed DEGHMOUM[1]*, Abderrahmane GHEZAL[2] and Said ABBOUDI[3]

[1]Département d'Energétique, Faculté des Sciences de l'Ingénieur, Université de M'Hammed Bougara, Boumerdes, UMBB 35000, Algérie.
[2]Institut de Physique, USTHB, B.P.32, El-Alia, Bab-Ezzouar, Alger, Algérie.
[3]IRTES-M3M, Université de Technologie de Belfort-Montbéliard, site de Sévenans, 90010, Belfort cedex, France.

Received 22 March, 2015; Accepted 8 September, 2015

This paper deals with the effects of magnetic field on heat transfer in a pulsatile flow. A mathematical model is developed to investigate the impact of magnetic field on the velocity and the temperature distributions between two concentric ducts. Finite differences method is used in order to solve the dimensionless governing equations, and implicit schemes for velocity and temperature are obtained. The effects of magnetic field on the velocity are represented by the Hartmann number. It is found that the increase of magnetic field leads to eliminate the annular effect of the pulsatile flow. It is also found that the velocity can be controlled by the external magnetic field which leads to affect the temperature profiles and so the heat transfer that could be improved or reduced by mastering the magnetic field.

Key words: Pulsatile flow, magnetohydrodynamics (MHD), concentric ducts, finite differences, blood flow.

INTRODUCTION

The study of pulsatile flow has been the subject of numerous investigations. The first work dates back to 1929 when Richardson and Tyler revealed by experimental measurements the existence of one of the main features of the oscillating flow which is called the annular effect. This effect is characterised by the presence of velocity maximums near the wall of the pipe. Later analyses of Womorsley (1955) and Uchida (1956) confirmed this result by analysing the sinusoidal motion of an incompressible fluid oscillating in a horizontal pipe. Atabek and Chang (1961) studied the unsteady flow in cylindrical pipe; they have developed an analytical solution for the velocity profile by assuming that the flow is established with far inputs.

Yakhot and Grinberg (2003) investigated the influence of the pressure gradient frequency on the velocity amplitude and the phase difference between the pressure gradient and the axial velocity. This phase difference varies from 0° for the slow frequencies to 90° for the high frequencies. Kakac and Yenner (1973) obtained an exact solution in the case of a forced flow between two parallel plates. Suces (1981) numerically investigated the response functions of the wall temperature and the average temperature between a laminar fluid flow and a flat plate by using a finite difference method. Zhao (1995) performed numerical and experimental studies on a

*Corresponding author. E-mail: mdeghmoum@hotmail.fr, Tel: 213 771 635 674. Fax: 213 21 24 73 44.

Author(s) agree that this article remain permanently open access under the terms of the Creative Commons Attribution License 4.0 International License

laminar air flow oscillating in a cylindrical pipe, heated by a uniform heat flux. From temperatures measured on several positions and at the inner wall of the heater, they obtained a correlation of average Nusselt number. Majdalani (2002) determined the exact solution of the Navier-Stokes equations governing the pulsatile flow in a cylindrical pipe where the pressure gradient was replaced by a sum of pulses expressed in terms of Fourier coefficients.

The application of magnetic field to a moving and electrically conducting liquid induces both electric and magnetic fields. A body force known as the Lorentz force is produced as a result of the interaction between the induced magnetic and electric fields. This force tends to oppose the movement of the liquid which leads to decrease the flow rate. Agrawal and Anwaruddin (1984) proposed a mathematical model for the effect of magnetic field on blood flow through an equally branched channel with flexible walls. They found that the magnetic field can be used as a blood pump in carrying out cardiac operations to cure some arterial diseases such as arteriosclerosis and arterial stenosis. Stud et al. (1977) examined the effect of a moving magnetic field on blood flow, and found that the application of a suitable magnetic field increases the blood flow rate.

In current study, we investigate analytically and numerically the effect of magnetic field on velocity and temperature distributions in case of pulsatile flow across a cylindrical duct. The importance of this study could be so sensible in the knowledge of blood behavior when subjected to a magnetic field and therefore offering best platform to reduce some arterial diseases. Finite differences method with an implicit scheme is used in order to solve the dimensionless governing equations. Velocity and temperature profiles are presented for different Womersley and Hartmann numbers.

MATHEMATICAL FORMULATIONS

Physical problem

The dynamic and thermal behaviors of a viscous and electrically conducting fluid flow between two cylindrical ducts, is presented in Figure 1. The fluid flow is subjected to a constant magnetic field and a pulsatile pressure gradient parallel to the axis.

$$\frac{\partial P}{\partial z} = -A\cos(\omega\, t)$$

Initially, the internal duct is at a temperature of Tint=400 K, the external duct is supposed adiabatic and the fluid is at a temperature of 300 K and atmospheric pressure.

Governing equations

Simplifying assumptions

1. The fluid is incompressible, viscous and electric conductor,
2. The flow is laminar and axisymmetric,
3. The energy losses due to viscosity are negligible,
4. The magnetic field is constant and radial.

Under the mentioned assumptions, the governing equations are:

Continuity equation:

$$\nabla\vec{U} = 0 \qquad (1)$$

Momentum equation:

$$\rho\left(\frac{\partial\vec{U}}{\partial t} + \vec{U}.\nabla\vec{U}\right) = -\nabla P + \mu\nabla^2\vec{U} + \vec{J}\wedge\vec{B} \qquad (2)$$

Energy equation:

$$\rho\, C_p\left(\frac{\partial T_f}{\partial t} + \left(\vec{U}.\nabla T_f\right)\right) = \lambda\nabla^2 T_f \qquad (3)$$

By introducing the following dimensionless variables:

$$r = \frac{r\prime}{R_e},\quad z = \frac{z\prime}{R_e},\quad t = \omega\, t',\quad w = \frac{w\prime}{\omega\, R_e},\quad T = \frac{T\prime - T_f}{T_i - T_f},$$
$$P = \frac{p\prime}{\rho\, R_e^2\, \omega^2},\qquad Re_\omega = \alpha^2 = \frac{R_e^2.\omega}{\nu},\qquad Pr = \frac{\mu.C_f}{k_f},$$
$$Ha = R_e\frac{H}{C}\sqrt{\frac{\sigma}{\nu.\rho}}.$$

The explicit form of the governing equations can be written as follows:

$$\frac{\partial u}{\partial r} + \frac{\partial w}{\partial z} + \frac{u}{r} = 0 \qquad (4)$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z} - \frac{v^2}{r} = \frac{\partial P}{\partial r} + \frac{1}{Re_\omega}\left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} - \frac{u}{r^2}\right] \qquad (5)$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z} = -\frac{\partial P}{\partial z} + \frac{1}{Re_\omega}\left[\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2}\right] - \frac{Ha}{Re_\omega}w \qquad (6)$$

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial r} + w\frac{\partial T}{\partial z} = \frac{1}{Re_\omega\, Pr}\left[\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}\right] \qquad (7)$$

Initial conditions:

t=0

$$u(r,z,0) = w(r,z,0) = 0$$
$$p(r,z,0) = T_f(r,z,0) = 0$$

Boundary conditions:

At the external duct:

$$u(1,z,t) = w(1,z,t) = 0$$
$$\frac{\partial T_f}{\partial r}(1,z,t) = 0$$

At the internal duct:

$$u\left(\frac{R_i}{R_e}, z, t\right) = w\left(\frac{R_i}{R_e}, z, t\right) = 0$$
$$T_f\left(\frac{R_i}{R_e}, z, t\right) = 1$$

ANALYTICAL SOLUTION

In order to solve the problem analytically, we assume that the flow is fully developed:

The governing equations of the fluid flow become:

Momentum equation:

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho}\frac{\partial P}{\partial z} + \vartheta\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)\right] - \frac{\sigma H^2}{\rho C^2} w \quad (8)$$

Energy equation:

$$\frac{\partial T(r,z,t)}{\partial t} + w\frac{\partial T}{\partial z} = \frac{k}{\rho C_p}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial z^2}\right] \quad (9)$$

Due to the axis-symmetry of the problem, the study can be reduced to the annular space between the two ducts.
The dimensionless equations (8) and (9) become:

$$\frac{\partial w}{\partial t} = -\frac{\partial P}{\partial z} + \frac{1}{\alpha^2}\left[\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right] - \frac{Ha^2}{\alpha^2} w \quad (10)$$

$$\frac{\partial T}{\partial t} + w\frac{\partial T}{\partial z} = \frac{1}{\alpha^2 . Pr}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial x^2}\right] \quad (11)$$

The pressure gradient can be written as:

$$\frac{\partial P}{\partial z} = -\tilde{A}.\mathrm{Cos}(\omega.t) = \mathrm{Re}\left(-\tilde{A}.e^{it}\right) \quad (12)$$

The velocity solution is sought in the form:

$$w(r,t) = Real\left(f(r).e^{i.t}\right) \quad (13)$$

By introducing this solution in equation (10), we obtain the following modified Bessel equation:

$$\frac{\partial^2}{\partial r^2}\bar{f} + \frac{1}{r}\frac{\partial}{\partial r}\bar{f} - [Ha^2 + i.\alpha^2]\bar{f} = -\tilde{A}\,\alpha^2 \quad (14)$$

Where the solution is a combination of Bessel functions I_0 and K_0 of first and second kind respectively:

$$f(r) = C_1 I_0(\eta r) + C_2 K_0(\eta r)$$

Where: $\eta = [Ha^2 + i\,\alpha^2]^{1/2}$

The constants C_1 and C_2 that are determined from the no slip boundary conditions:

$$r = \frac{R_i}{R_e} \quad w = 0 \quad (15)$$

$$r = 1 \quad w = 0 \quad (16)$$

In order to ease the form of equations, we pose:

$$R_i^* = \frac{R_i}{R_e}$$

Therefore:

$$C_1 = \left[\frac{\frac{\tilde{A}^*.\alpha^2}{\eta^2}\left(K_0(\eta R_i^*) - K_0(\eta)\right)}{I_0(\eta R_i^*).K_0(\eta) - K_0(\eta R_i^*).I_0(\eta)}\right] \quad (17)$$

and

$$C_2 = \left[\frac{\frac{\tilde{A}^*.\alpha^2}{\eta^2}\left(I_0(\eta) - I_0(\eta.R_i^*)\right)}{I_0(\eta R_i^*).K_0(\eta) - K_0(\eta R_i^*).I_0(\eta)}\right] \quad (18)$$

Thus, the evolution of the velocity profile can be written as follows:

$$w(r,t) = Real\left\{C_1 I_0(\eta r) + C_2 K_0(\eta r) + \frac{\tilde{A}^*\alpha^2}{\eta^2}\right\}e^{it} \quad (19)$$

In order to solve analytically the equation (11), we assume that the temperature solution profile can be written as:

$$T_f^*(r,z,t) = Real\left(-\gamma^*.z + \gamma^* g(r)e^{it} + 1\right) \quad (20)$$

With: $\gamma^* = \frac{R_e}{L}$

Therefore, we obtain the following differential equation:

$$\frac{\partial^2}{\partial r^2} g + \frac{1}{r}\frac{\partial}{\partial r} g - i.\alpha^2 Pr.g = -\alpha^2.Pr.f \quad (21)$$

By using the following boundary conditions:

$$r = R_i^* \Rightarrow T = 1 \quad (22)$$

$$r = 1 \Rightarrow \frac{\partial T}{\partial r} = 0 \quad (23)$$

The temperature solution profile can be written as:

$$T(r,z,t) = Real\left\{-\gamma^* z + \gamma^*\left[A_1 I_0(\delta r) + A_2 K_0(\delta r) - iC_1 I_0(\eta r) - iC_2 K_0(\eta r) - \frac{i\tilde{A}\alpha^2}{\eta^2}\right]e^{it} + 1\right\} \quad (24)$$

Where: $\delta = \alpha\sqrt{i.Pr}$

$$A_1 = \left[\frac{\left(\psi_1.\delta.K_1(\delta) + \psi_2 K_0(\delta R_i^*)\right)}{I_0(\delta R_i^*).\delta.K_1(\delta) + K_0(\delta.R_i^*).\delta.I_1(\delta)}\right]$$

$$A_2 = \left[\frac{\left(\psi_1.\delta.I_1(\delta) - \psi_2 I_0(\delta R_i^*)\right)}{I_0(\delta R_i^*).\delta.K_1(\delta) + K_0(\delta.R_i^*).\delta.I_1(\delta)}\right]$$

$$\psi_1 = i.\left[C_1 I_0(\eta.R_i^*) + C_2 K_0(\eta.R_i^*) + \frac{\tilde{A}^*.\alpha^2}{\eta^2}\right] + \frac{x}{e^{i\,t}}$$

$$\psi_2 = i.\left[C_1.\eta.I_1(\eta) - C_2.\eta.K_1(\eta)\right]$$

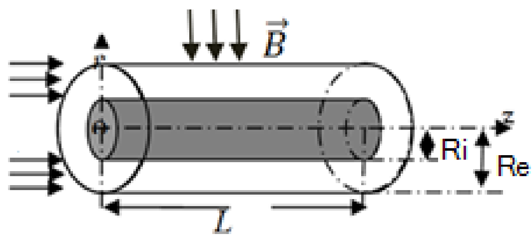

Figure 1. Flow field geometry.

NUMERICAL ANALYSIS

The system of Equation (4) to (7) with the corresponding initial and boundary conditions is solved numerically by finite differences method using implicit scheme. The obtained solution at the fully developed regime will be compared to the analytical solution (19) for the axial velocity and (24) for the temperature.

At each new time, the system of the algebraic equations resulting from the FDM discretization have tri-diagonal matrix form which is solved by TDMA Algorithm.

Because the problem of this study is axisymmetric, the computational domain is reduced to the mesh grid domain illustrated in Figure 2a.

In the vicinity of the ducts, the mesh is refined by replacing the mesh situated near the wall of the internal duct by sub decreasing mesh size following geometric sequences of G (Figure 2b). Where the sum of sub-mesh sizes is equal to the size of a mesh grid. Other meshes that are far from the ducts remain the same size.

RESULTS AND DISCUSSION

Figures 3 to 8 show the analytical and numerical solutions obtained for the velocity profiles for $\frac{R_i}{R_e} = 0.3$ and Ha=0.

It can be seen that there is a large similarity between the analytical and numerical results, which validate the numerical method used in this study. Some differences exist because the analytic solution takes into consideration one-directionality of the problem.

In order to show the effect of magnetic field on the velocity profiles, the following results are shown in Figures 9 to 11 without magnetic field (Ha=0) for $\frac{R_i}{R_e} = 0.3$ and different Womersley (Re_ω).

However, the following results that are shown in Figures 12 to 14 illustrate the effect of magnetic field on the velocity profiles for a wide range of Hartmann numbers (Ha=1, 15, 30).

This results show that the maximum of velocity in a pulsatile flow is situated near the walls of the ducts, which is called the annular effect, revealed by experimentally by Richardson and others and developed analytically by Atabek and Chang (1961). This annular effect increases by the increase of Womersley number Re_ω. The Figure 15 shows the influence of Womersley number on the situation of velocity maximums in the annular space between the two ducts.

In the other hand, it can be seen from Figures 12 to 14 that the magnetic field acts as a retardant against the flow which leads to decrease the flow rate. Furthermore, the magnetic field leads to eliminate the annular effect which is considered as a characteristic of the pulsatile flow.

The flow area also influences on the velocity profiles by eliminating the annular effect as it is shown in (Table 1) where the dimensionless radius of the internal duct varies from 0.3 to 0.8

The results show that the decrease in the flow area from 0.7 to 0.5 leads to decrease gradually the annular effect. Nevertheless, the annular effect is almost absent when the flow area is in the vicinity of 0.4 to 0.2.

Figure 16 illustrates the influence of flow area reduction on velocity profile and its impact on the annular effect. The reduction of flow area leads to increase the velocity as it is shown in Figure 16, where the velocity increases from about 0.4 to 1.2 for the same Re_ω, t and Ha. However, it can be seen that the velocity maximums are in the vicinity of the ducts for the flow areas that vary from 0.7 to 0.5 and the reduction in the flow area leads to reduce the annular effect until its disappearance for the flow areas that vary from 0.4 to 0.2 where the velocity maximum is situated in the middle of the annular space.

Figures 17 and 18 shows a comparison between results of the vortex profiles obtained by the present study and those obtained by Majdalani (2008).

It can be seen that the results of vortex profiles obtained by the present study resemble to the results obtained by Majdalani (2008). However, the existence of slight differences is due to the fact that Majdalani (2008) worked on a pulsatile flow in a rectangular duct.

The Figure 19 shows the temperature profiles for a moderate flow regime (Re_ω=10) and without the application of magnetic field (Ha=0). However, the Figure 20 shows the temperature profiles for the same regime but in presence of magnetic field (Ha=15).

At the light of the results presented in Figures 19 and 20, it appears that the application of an external magnetic field has improved the heat transfer between the two ducts. In addition, the velocity can be controlled by managing the magnetic field which means that the heat transfer can be reduced or enhanced depending on the application required.

Conclusion

The effect of magnetic field on heat transfer has been studied analytically and numerically. An exact solution for velocity and temperature distribution across the annular space between two cylinders in case of a pulsatile flow is developed, which is precious for the knowledge of blood behavior when subjected to a magnetic field which will lead to further improve the reduction of some arterial

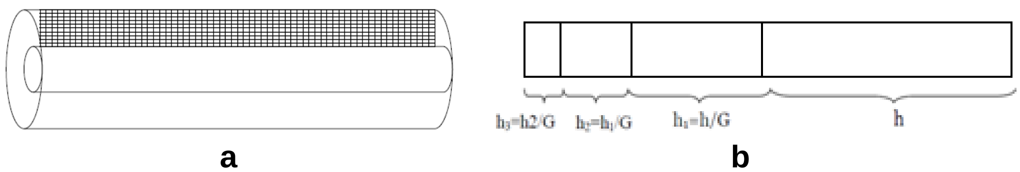

Figure 2. (a) Computational domain, (b) Scheme of the grid independency analysis.

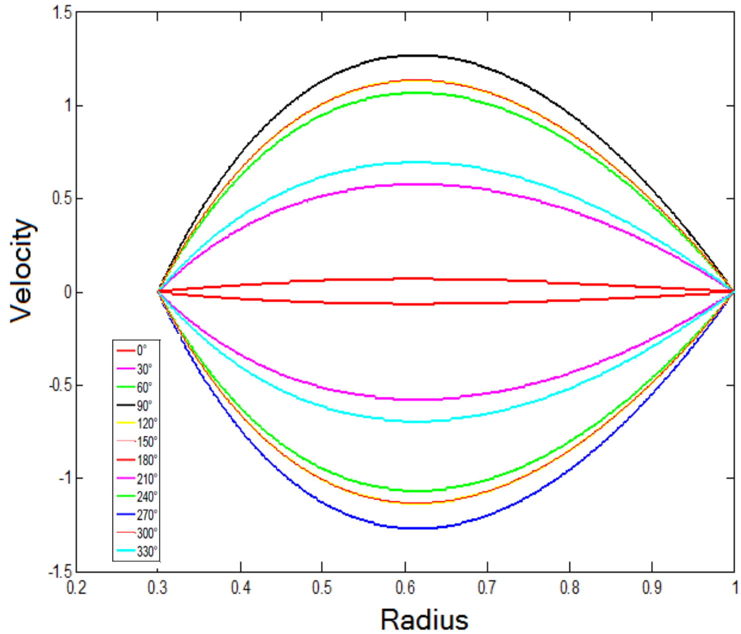

Figure 3. Analytical velocity profiles for Reω=1.

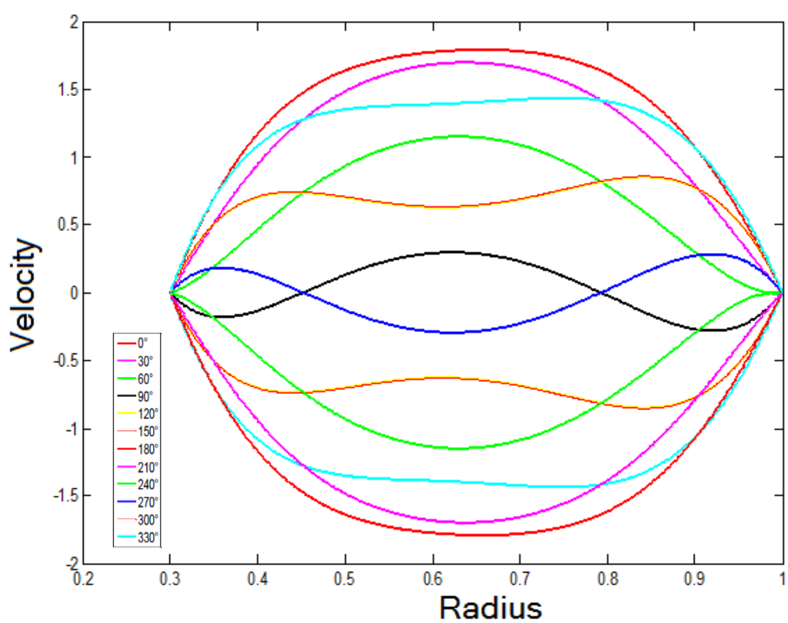

Figure 4. Analytical velocity profiles for Re_{ω}=10.

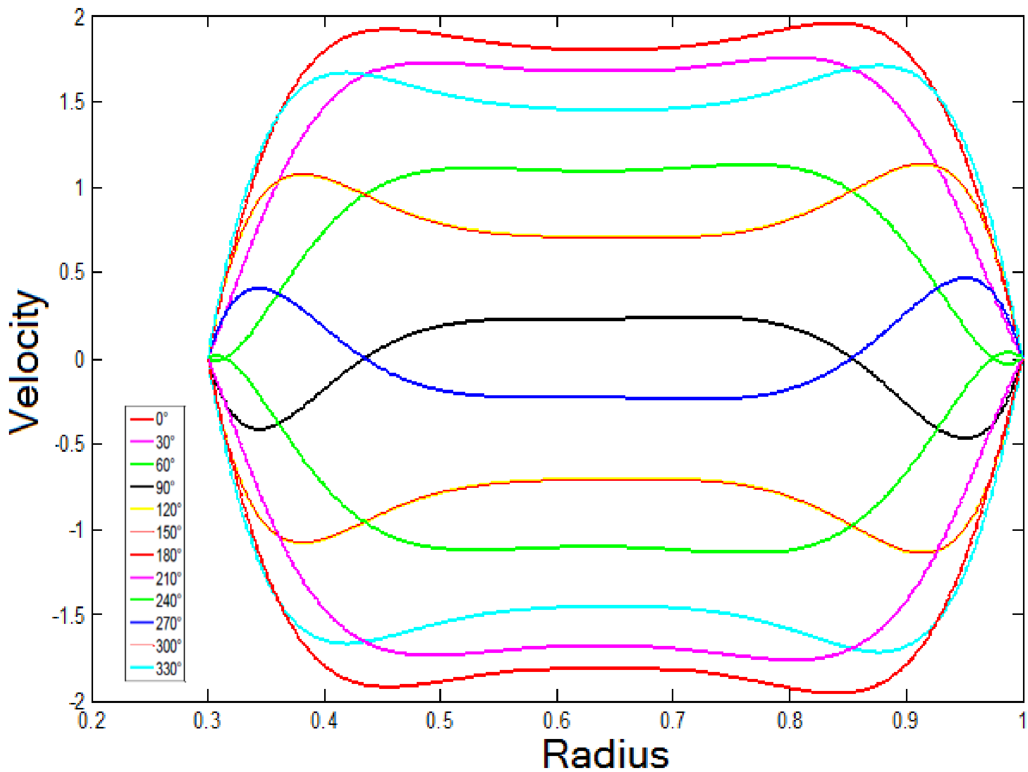

Figure 5. Analytical velocity profiles for Re_{ω}=30.

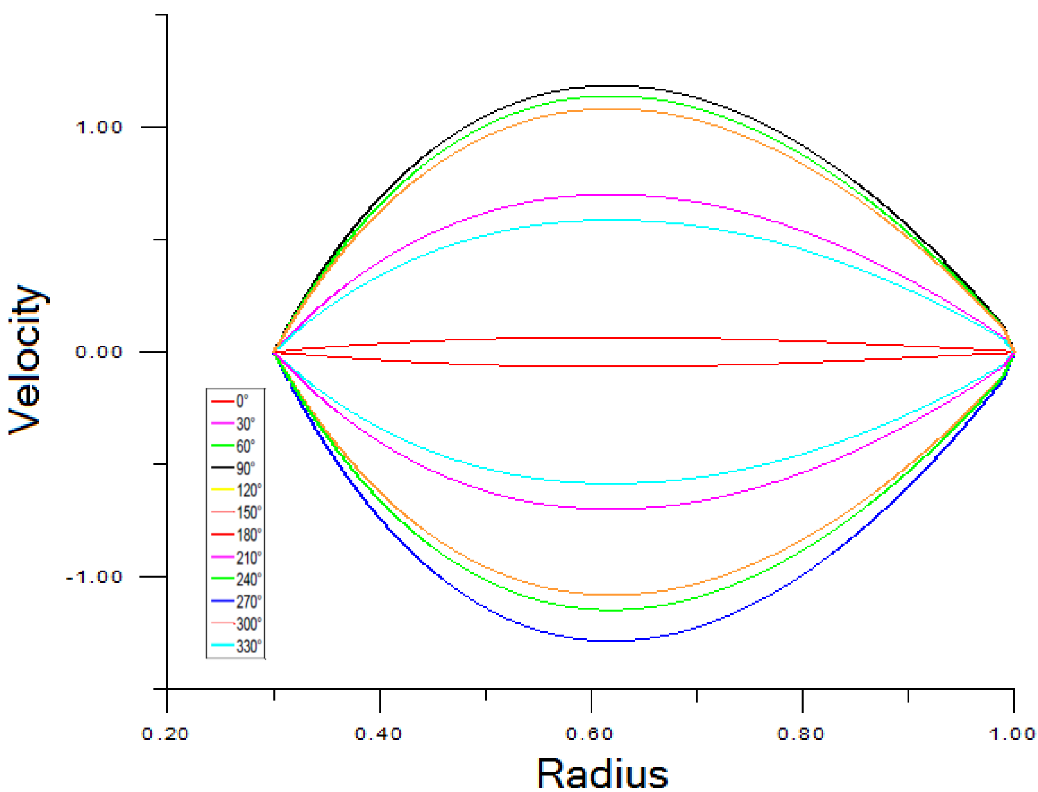

Figure 6. Numerical velocity profiles for Re_{ω}=1.

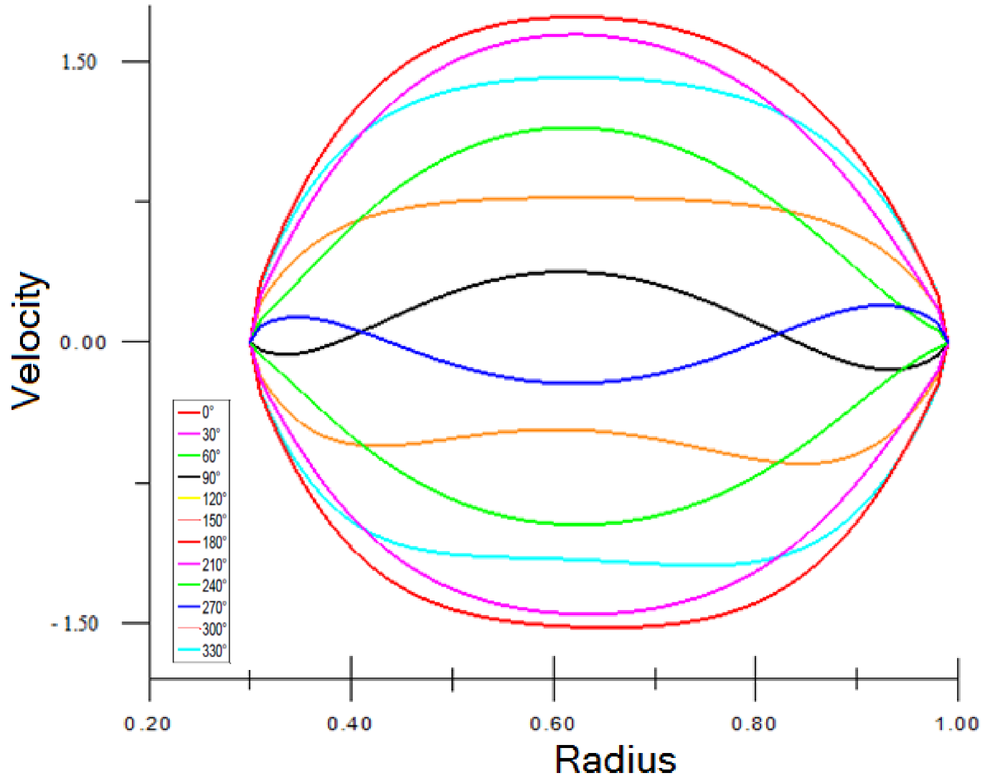

Figure 7. Numerical velocity profiles for Re_{ω}=10.

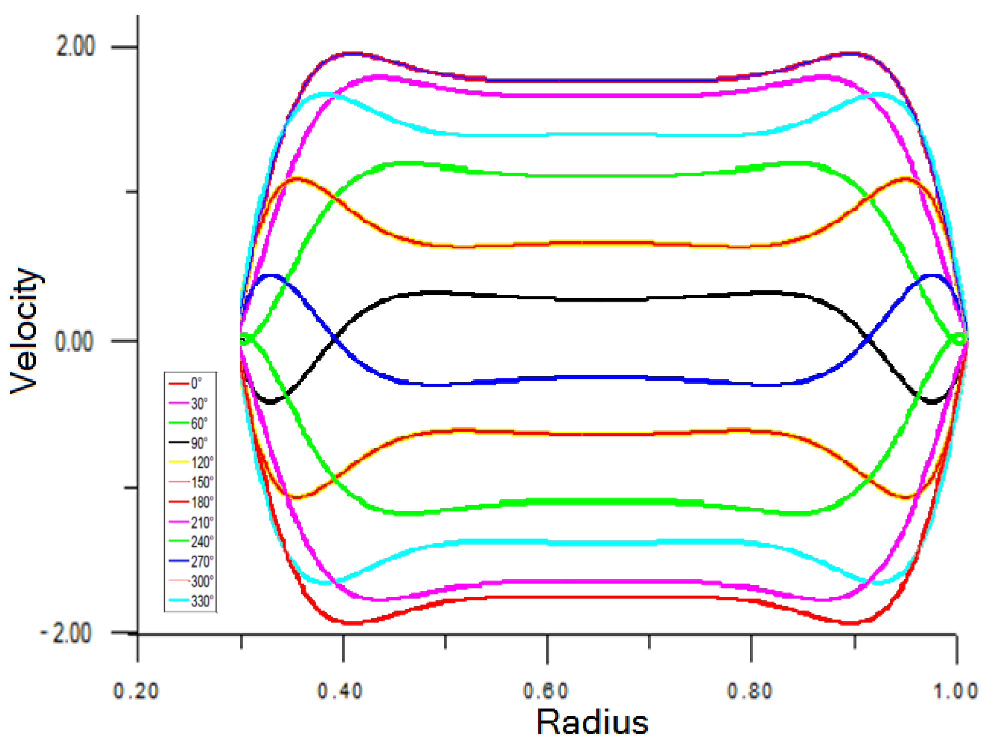

Figure 8. Numerical velocity profiles for Re_{ω}=30.

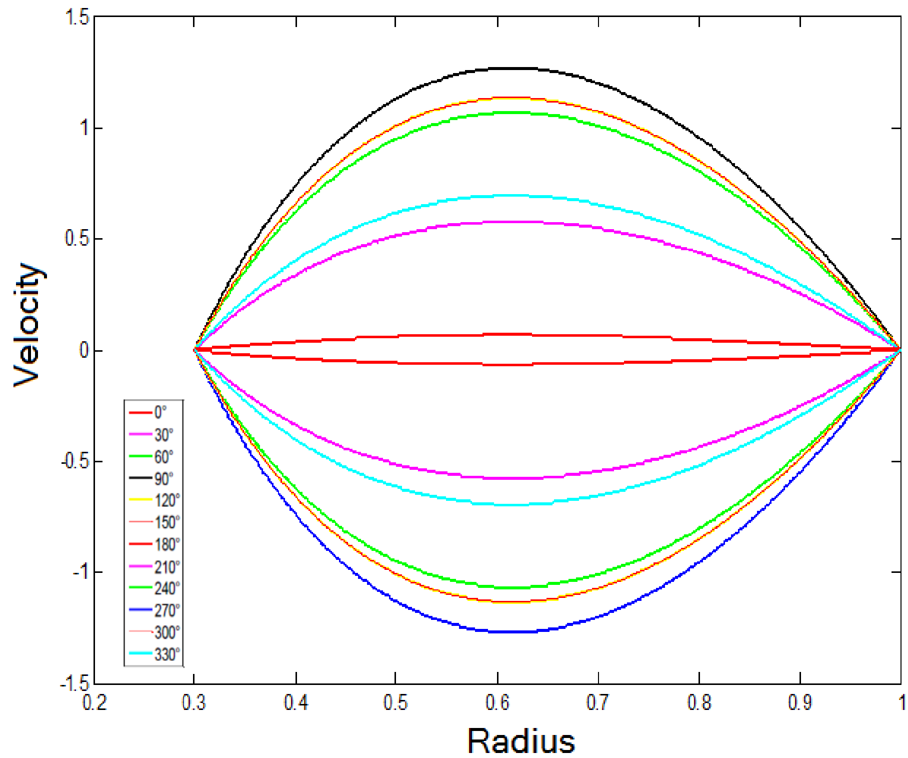

Figure 9. Velocity profiles for Re_{ω}=1, Ha=0.

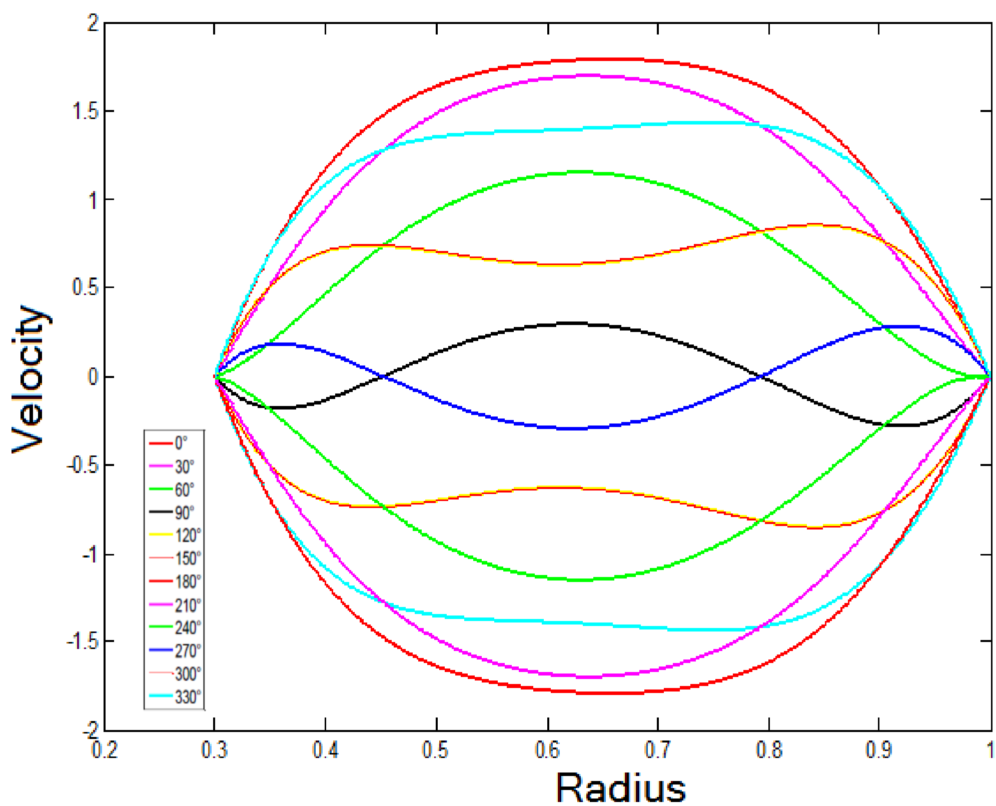

Figure 10. Velocity profiles for Re_{ω}=10, Ha=0.

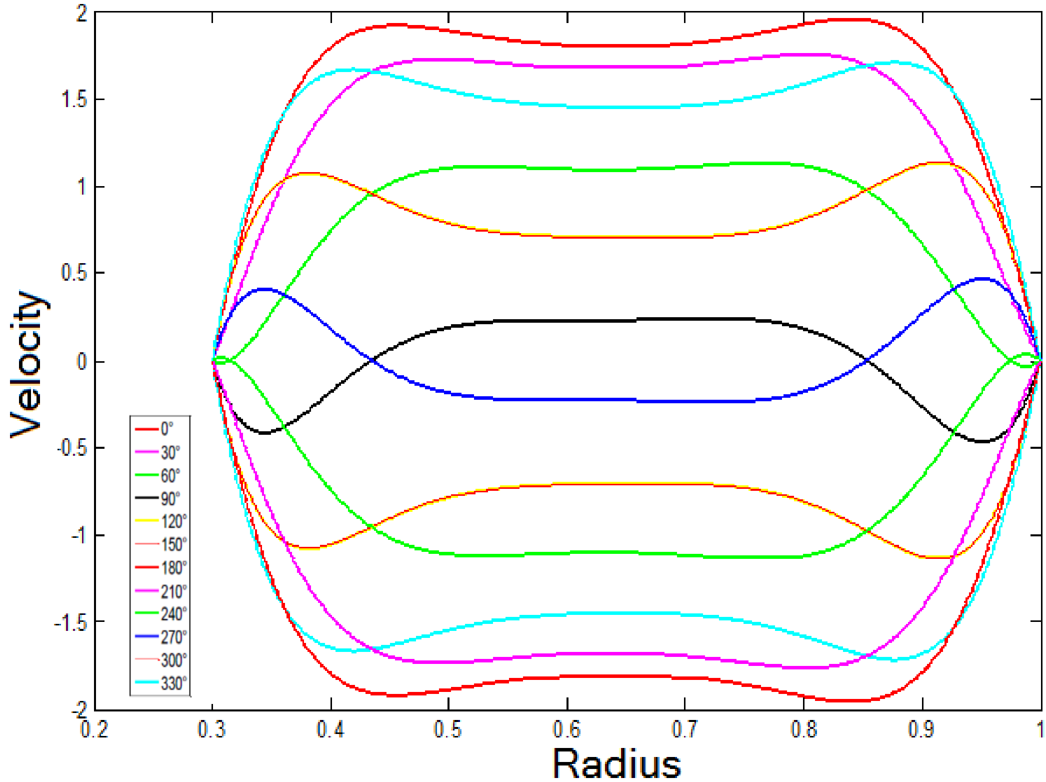

Figure 11. Velocity profiles for Re□=30, Ha=0.

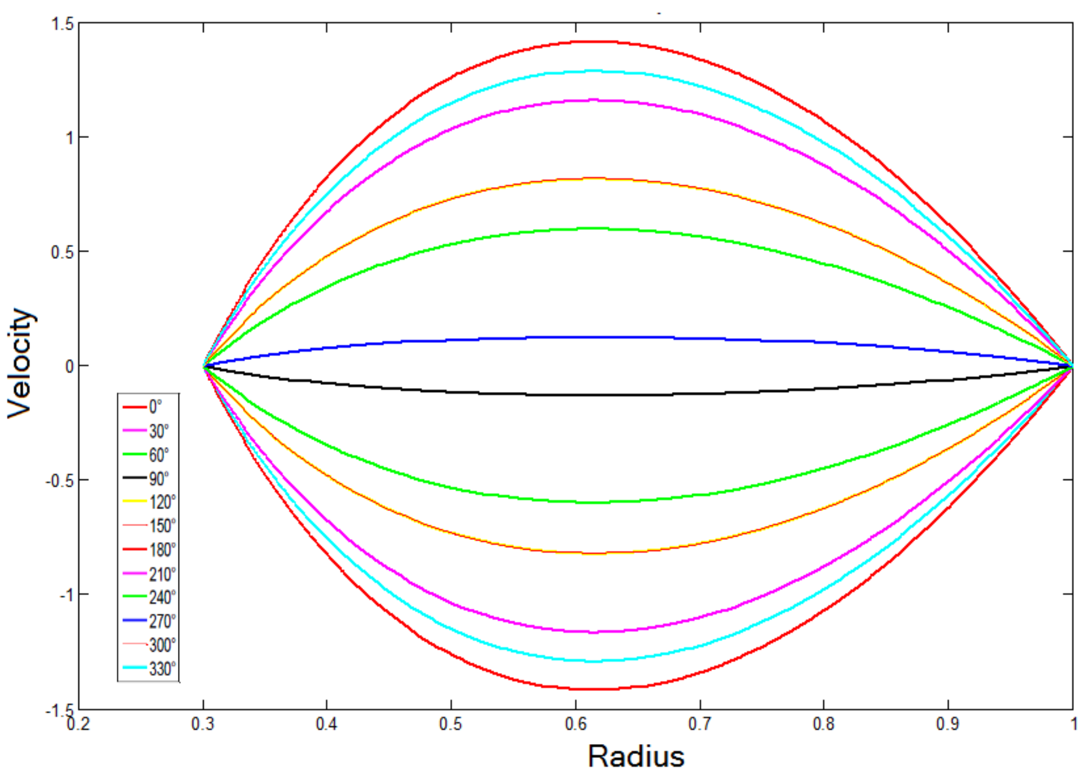

Figure 12. Velocity profiles for Re_{ω}=1, Ha=1.

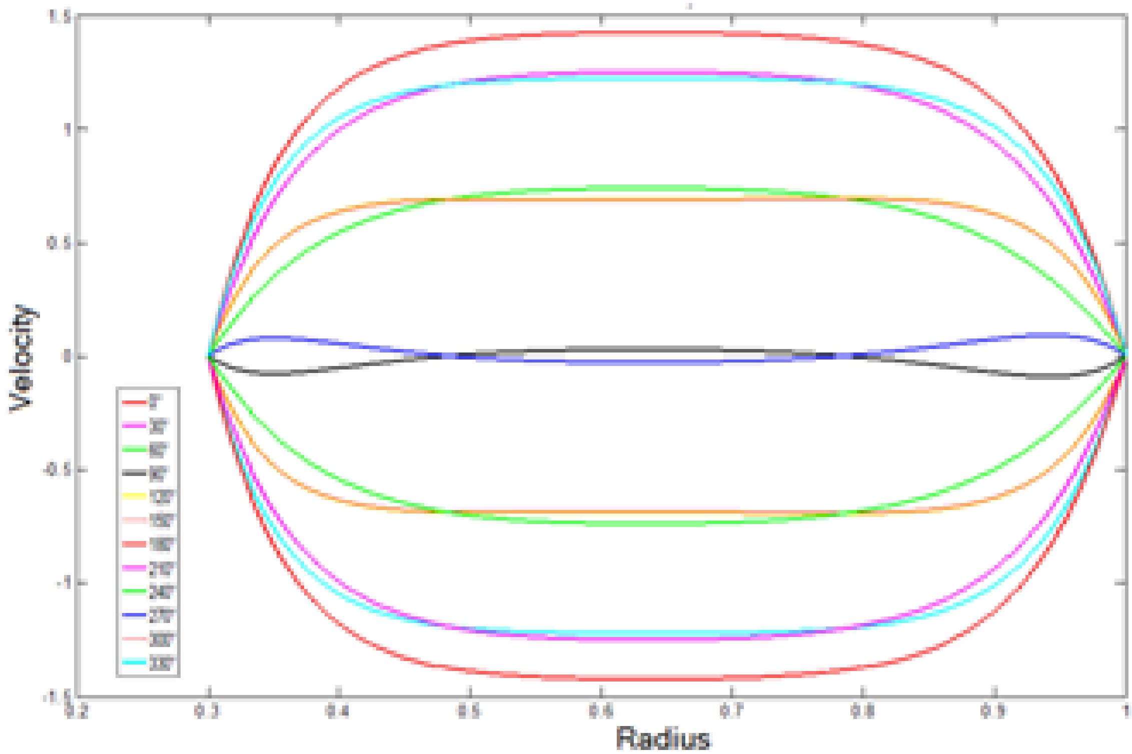

Figure 13. Velocity profiles for Re_{ω}=10, Ha=15.

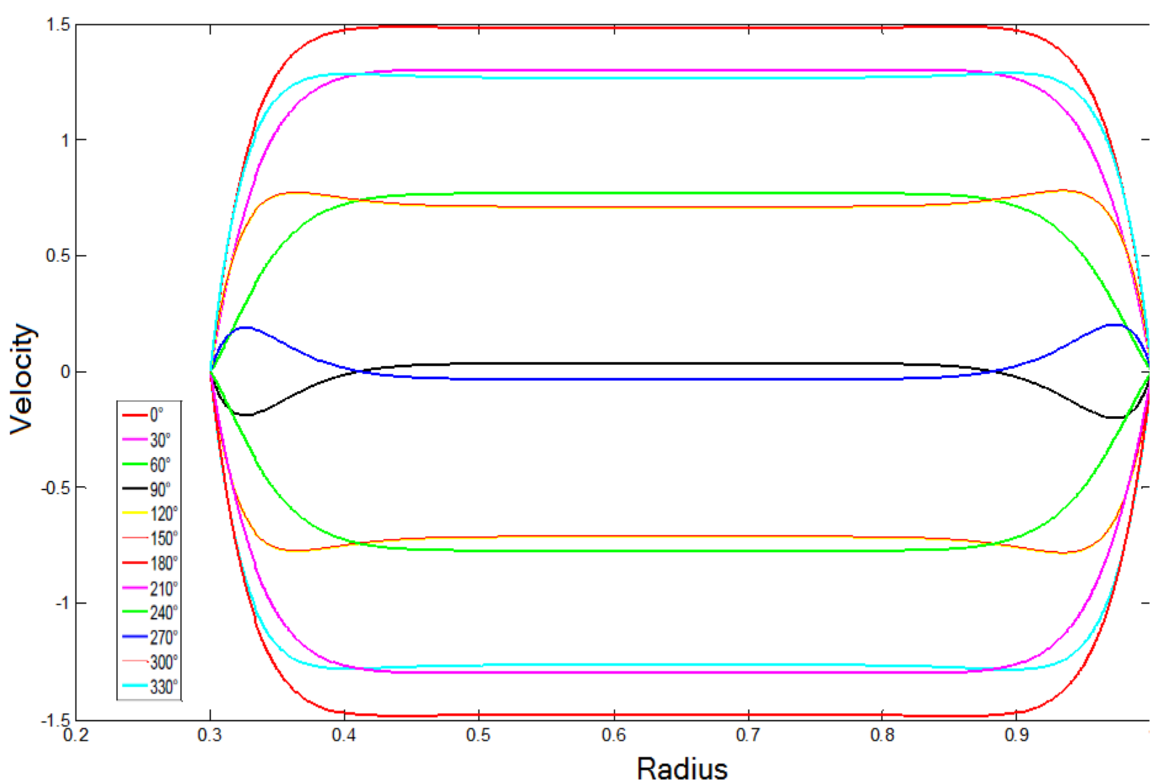

Figure 14. Velocity profiles for Re□=30, Ha=30.

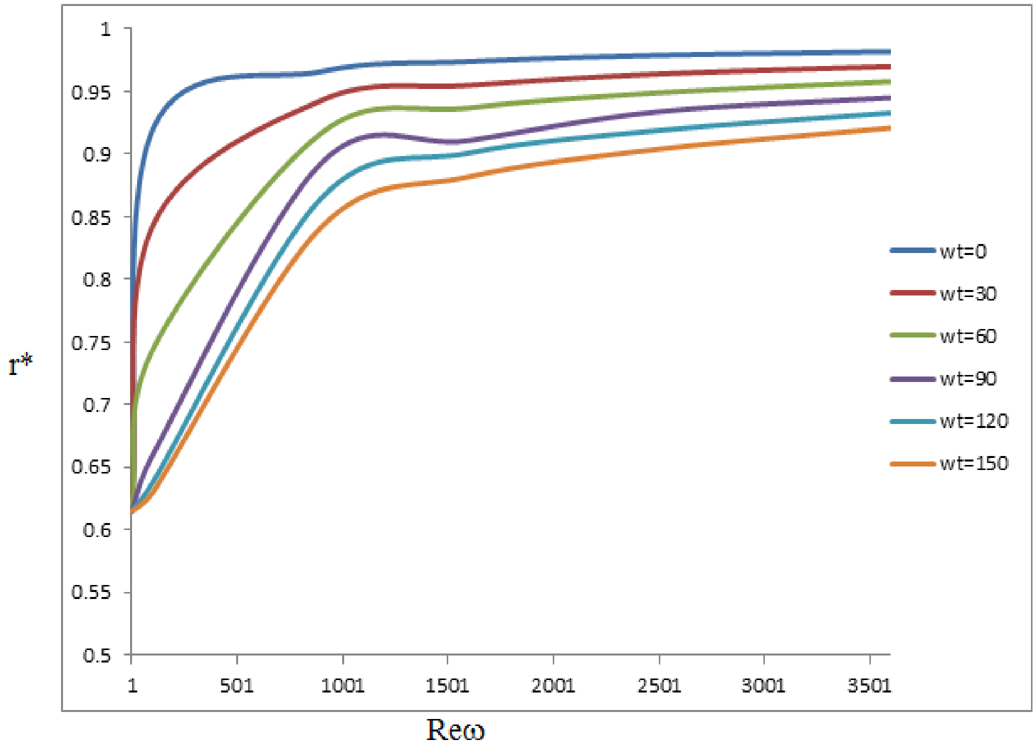

Figure 15. Position of velocity maximums for different Womersley numbers and Ha=0.

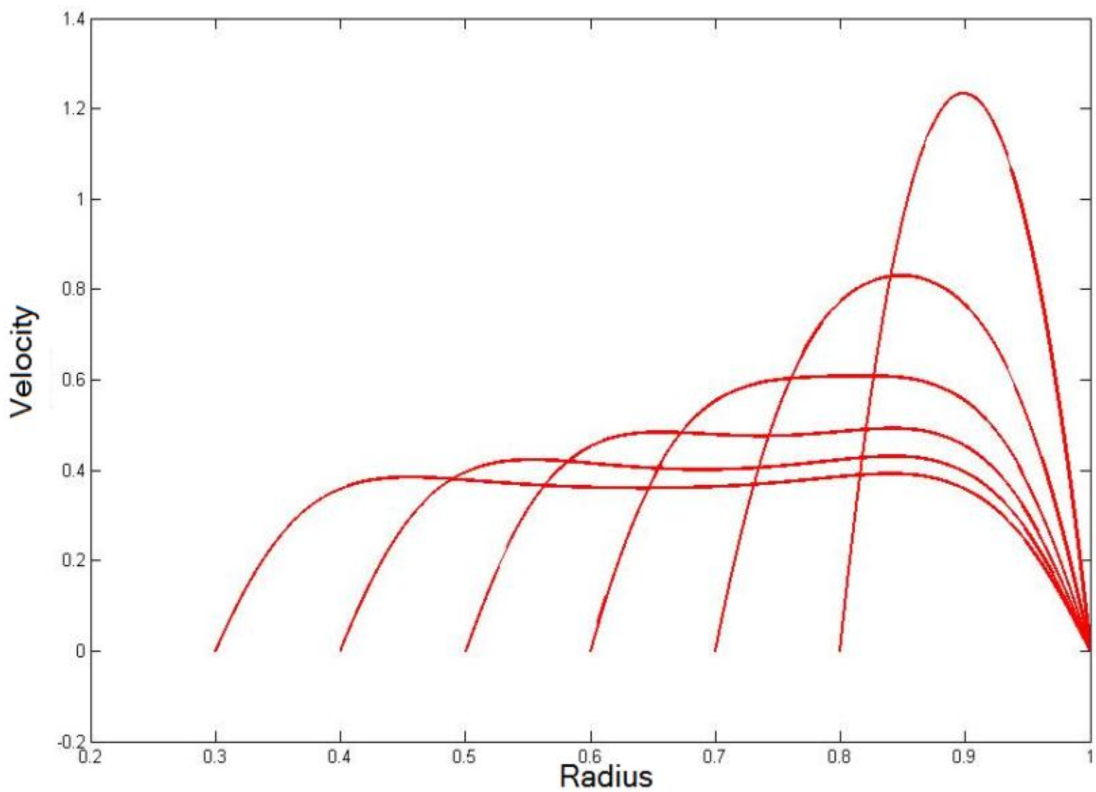

Figure 16. Influence of the flow area (from 0.3 to 0.8) on the velocity profiles for Re_{ω}=20, t=30^{o}, Ha=0.

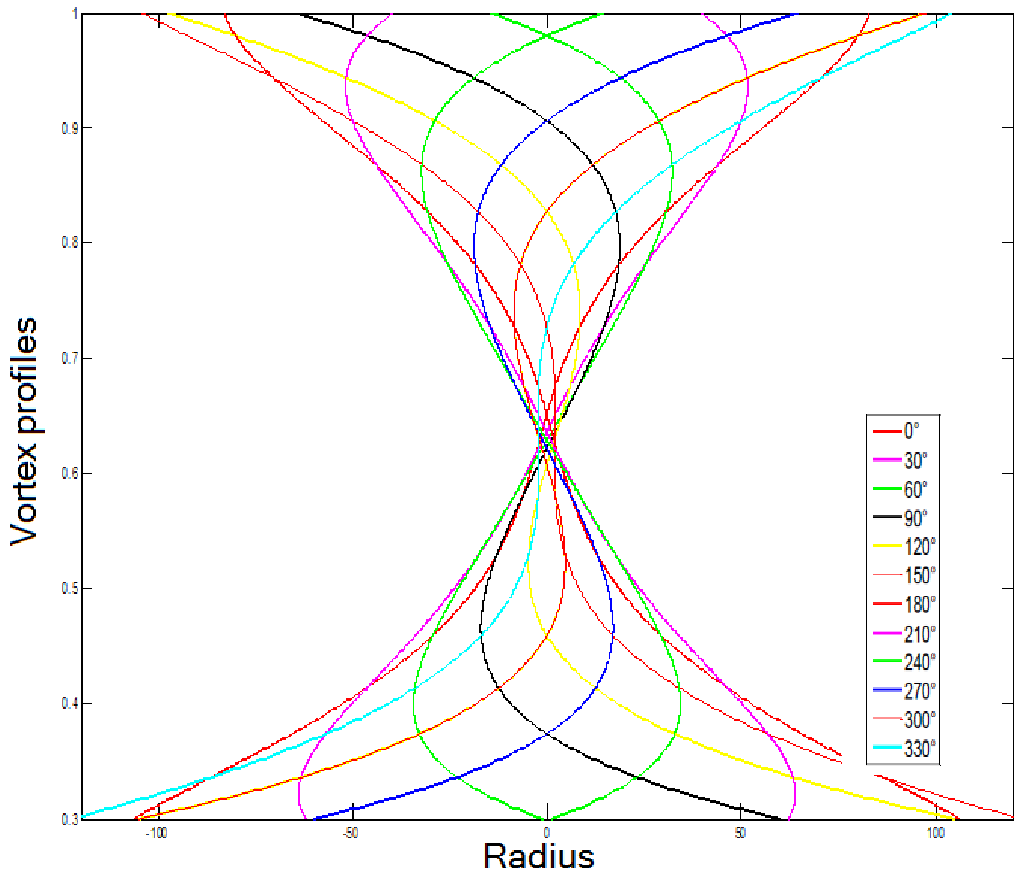

Figure 17. Vortex profiles for Re_{ω}=10 obtained by the present study.

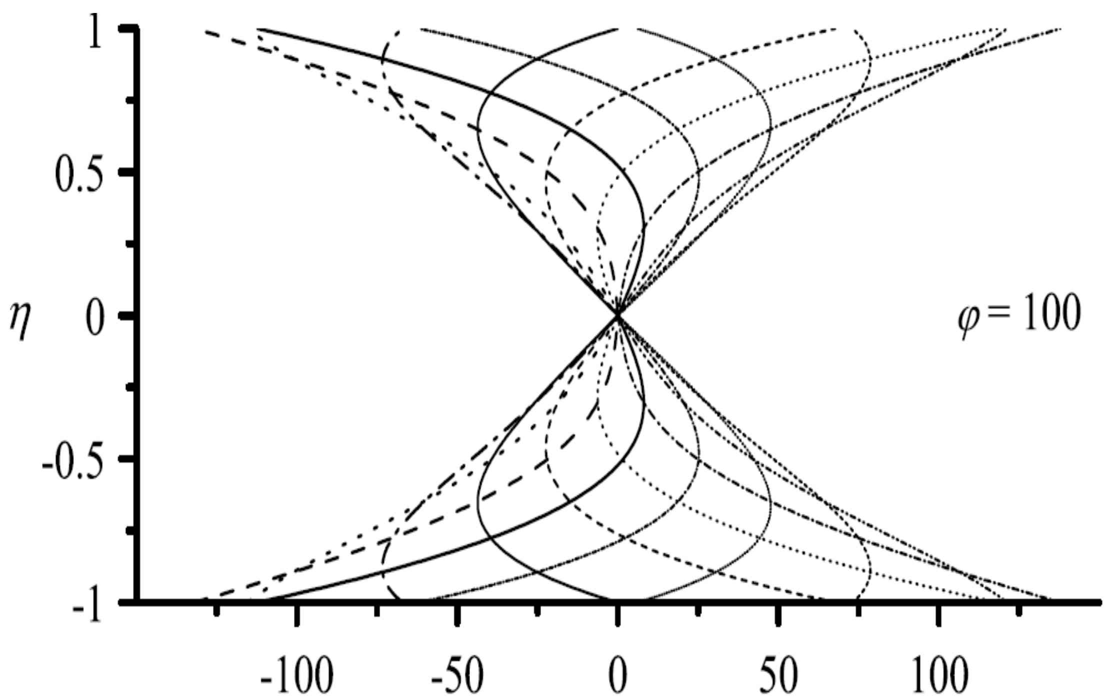

Figure 18. Vortex profiles for Re_{ω}=10 obtained by Majdalani (2008).

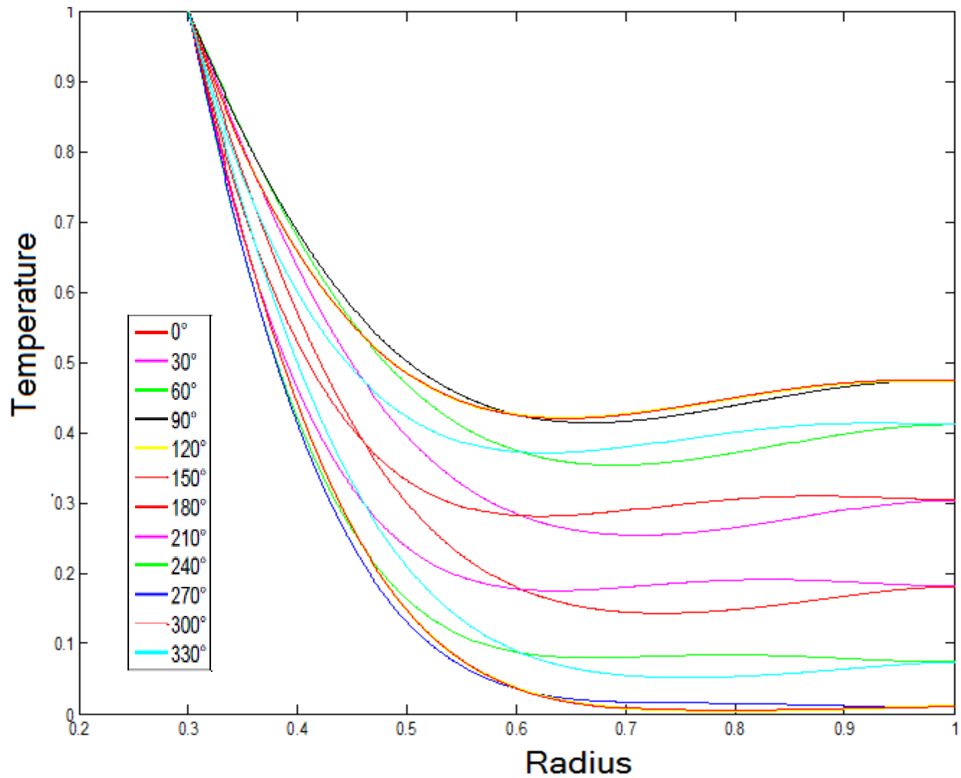

Figure 19. Temperature profiles for Re□=10 and Ha=0.

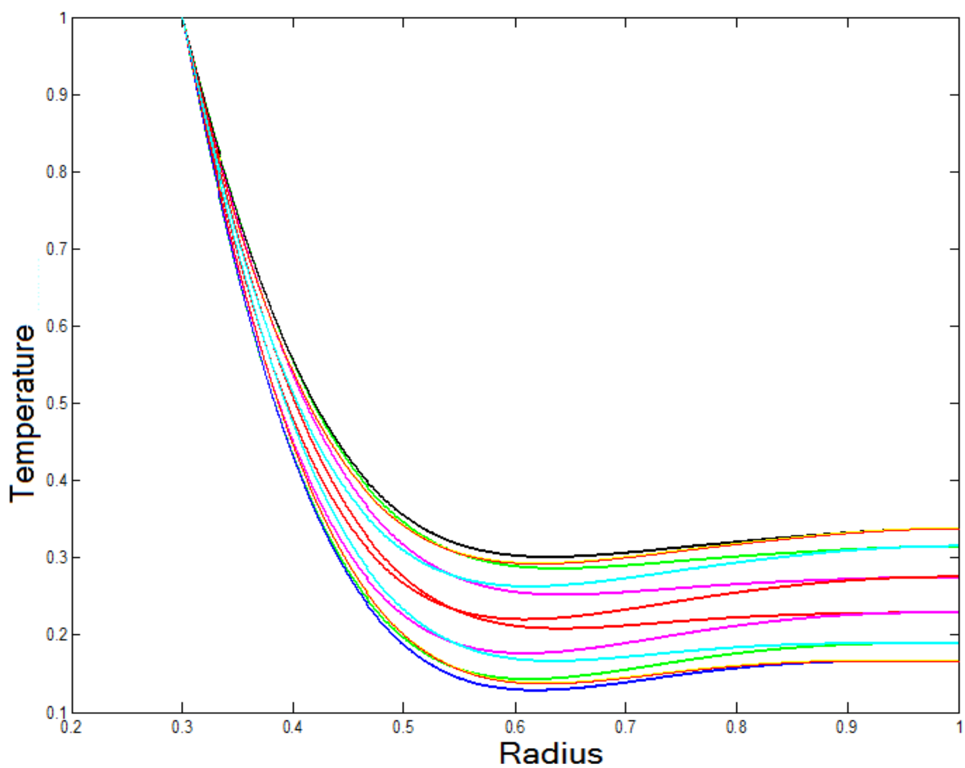

Figure 20. Temperature profiles for Re_{ω}=10 and Ha=15.

Table 1. Position of velocity maximums for Re□=20, $t=30^{o}$, Ha=0.

Dimensionless radius	Position of velocity maximum	Percentage
0.3	0.455	22.14
0.4	0.551	25.17
0.5	0.663	32.6
0.6	0.791	47.75
0.7	0.849	49.66
0.8	0.900	50.00

diseases. The developed analytical solutions for the velocity and temperature are shown graphically for a wide range of Womersley and Hartmann numbers. The results showed that the constant magnetic field imposed to the pulsatile flow leads to eliminate the annular effect, which is a characteristic of this type of flow. Furthermore, the results showed that the velocity could be controlled by the external magnetic field and also the temperature and so the heat transfer could be reduced or improved by mastering the intensity of the magnetic field.

Conflict of Interest

The authors have not declared any conflict of interest.

REFERENCES

Agrawal HL, Anwaruddin B (1984). Peristalic flow of blood in a branch. Ranchi Univ. Math. J. 15:111-121.

Atabek HB, Chang CC (1961). Oscillatory flow near the entry of circular tube. Zamp 12:405-422.

Kakac S, Yenner Y (1973). Exact solutions of transient forced convection energy equation for time with variation of inlet temperature. Int. J. Heat Mass. Trans. 11:2205.

Majdalani J (2002). Pulsatory channel flows with arbitrary pressure gradients. AIAA.3rd. Theoretical fluid mechanics meeting. pp. 24-26.

Majdalani J (2008). Exact Navier-stokes solution for pulsatory viscous channel flows with arbitrary pressure gradient. J. Propul. Power 24(6).

Stud VK, Sephon GS, Mishra RK (1977). Pumping action on blood flow by a magnetic field. Bull. Math. Biol. 39:385-390.

Suces J (1981). An improved quasi-study approach for transient conjugated forced convection problems. Int. J. Heat. Mass. Trans. 24(10):1711-1722.

Yakhot A, Grinberg L (2003). Phase shift ellipses for pulsating flows. Phys. Fluids 15(7).

Zhao T (1995). A numerical Solution of laminar flow convection in a heated pipe subjected to a reciprocating flow. Int. J. Heat Mass Trans. 38(16):3011-3022.

Womersley JR (1955). Oscillatory motion of viscous liquid in a thin walled elastic tube: The linear approximation for long waves, Phill. Mag. 46(7):199-221.

Uchida S (1956). The pulsating viscous flow superposed on the steady laminar motion of incompressible fluid in a circular pipe. Zeitschrift für angewandte Mathematik und Physik ZAMP. 7(5):403-422.

academicJournals

Vol. 10(24), pp. 604-613, 30 December, 2015
DOI: 10.5897/IJPS2015.4384
Article Number: 7ED59C856727
ISSN 1992 -1950
Copyright ©2015
Author(s) retain the copyright of this article
http://www.academicjournals.org/IJPS

International Journal of Physical Sciences

Full Length Research Paper

Gravitational effect on surface waves in a homogeneous fibre-reinforced anisotropic general viscoelastic media of higher and fractional order with voids

A. Khan[1]*, S. M. Abo-Dahab[2,3] and A. M. Abd-Alla[2,4]

[1]Department of Mathematics, COMSATS, Institute of Information, Park Road, Chakshahzad, Islamabad, Pakistan.
[2]Mathematics Department, Faculty of Science, Taif University 888, Saudi Arabia.
[3]Mathematics Department, Faculty of Science, SVU, Qena 83523, Egypt.
[4]Mathematics Department, Faculty of Science, Sohag University, Egypt.

Received 25 June, 2015; Accepted 10 October, 2015

In this paper, gravitational effects on propagation of surface waves in a homogeneous fibre-reinforced anisotropic general viscoelastic media of higher order with voids is investigated. The general surface wave speed is derived to study the effects of gravity on surface waves. Particular cases for Stoneley and Rayleigh waves are discussed. The results obtained in this investigation are more general in the sense that some earlier published results are obtained from our result as special cases. In the absence of voids our results for viscoelastic of order zero are well agreement to fibre-reinforced materials. Also by neglecting the reinforced elastic parameters, the results reduce to well known isotropic medium. Numerical results for particular materials are given and illustrated graphically. The results indicate that the effect of the gravitational, voids and the reinforced elastic parameters on surface waves are very pronounced.

Key words. Fibre-reinforced, viscoelastic, surface waves, gravity, anisotropic, voids.

INTRODUCTION

It is of great interest to study the propagation of surface waves in a homogeneous fibre-reinforced anisotropic general viscoelastic media of higher order with voids as it plays an importent role in material fracture and failure. Such problems have attracted much attention and have undergone a certain development (Bullen, 1965; Ewing and Jardetzky, 1957; Rayleigh, 1885; Stoneley, 1924). Surface waves have been well recognized in the study of earthquake, seismology, geophysics and geodynamics. These waves usually have greater amplitudes as compared with body waves and travel more slowly than body waves. There are many types of surface waves but we only discussed Stoneley and Rayleigh waves. In earthquake the movement is due to the surface waves.

*Corresponding author. E-mail: aftab.khan@comsats.edu.pk.

Author(s) agree that this article remain permanently open access under the terms of the Creative Commons Attribution License 4.0 International License

These are also used for detecting cracks and other defects in materials. Lord Rayleigh (1885) was the first to observe such kind of waves in 1885. That is why we called it Rayleigh waves. Sengupta and Nath (2001) investigated surface waves in fibre-reinforced anisotropic elastic media but their decomposition of displacement vector was not correct due to which some errors are found in their investigations (Sarvajit, 2002).

The idea of continuous self-reinforcement at every point of an elastic solid was introduced by Belfield et al. (1983). The superiority of fibre-reinforced composite materials over other structural materials attracted many authors to study different type of problems in this field. Fibre-reinforced composite structures are used due to their low weight and high strength. Two important components namely concrete and steel of a reinforced medium are bound together as a single unit so that there can be no relative displacement between them, that is, they act together as a single anisotropic unit. The artificial structures on the surface of the earth are excited during an earthquake, which give rise to violent vibrations in some cases. Engineers and architects are in search of such reinforced elastic materials for the structures that resist the oscillatory vibration. The propagation of waves depends upon the ground vibration and the physical properties of the structure material. Surface wave propagation in fiber reinforced media was discussed by various authors.

In classical theory of elasticity, the voids is an important generalization. Nunziato and Cowin (1979) and Cowin and Nunziato (1983) discussed the theory in elastic media with voids. Puri and Cowin (1985) studied the effects of voids on plane waves in linear elastic media and it is evident that pure shear waves remain unaffected by the presence of pores. Theory of thermoelastic material with voids is investigated by Lesan (1986). Good amount of literature on surface wave propagation in a generalized thermoelastic material with voids, is available in Singh and Pal (2011) and references therein. Chandrasekharaiah (1987a, b) discussed the effects of voids on propagation of plane and surface waves. Abo-Dahab (2010) investigated the propagation of P waves from stress-free surface elastic half-space with voids.

The effect of gravity on wave propagation in an elastic solid medium was first considered by Bromwich (1898). Later on gravity effects on wave propagation were discussed by various authors (Abd-Alla et al., 2013; Abd-Alla and Ahmed, 2003; De and Sengupta, 1974; Sengupta and Acharya, 1979)

Surface waves in fiber-reinforced,general viscoelastic media of higher order under gravity is discussed by kakar et. al. (2013) whereas Pal and Sengupta (1987) studied the gravitational effects in viscoelastic media. Ren et al. (2012) investigated the coupling effects of void shape and void size on the growth of an elliptic void in a fiber-reinforced hyper-elastic thin plate. Vishwakarma et al. (2013) discussed the influence of rigid boundary on the love wave propagation in elastic layer with void pores. Tvergaard (2011) studied the elastic–plastic void expansion in near-self-similar shapes. Fonseca et al. (2011) expressed the material voids in elastic solids with anisotropic surface energies. The extensive literature on the topic is now available and we can only mention a few recent interesting investigations in Abo-Dahab and Abd-Alla (2014), Abd-Alla et al. (2011), Abd-Alla and Ahmed (2003), Abd-Alla (1999), Abd-Alla and Ahmed (1999), Abd-Alla et al. (2004), Elnaggar and Abd-Alla (1989), Abd-Alla and Ahmed (1996) Abd-Alla et al. (2012) and Abd-Alla et al. (2013). Aim of this paper is to investigate the gravitational effects on propagation of surface waves in fibre-reinforced viscoelastic anisotropic media of higher order with voids. The general surface wave speed is derived to study the effect of gravity and voids on surface waves. Particular cases for Stonely and Rayleigh waves are discussed. The results obtained in this investigation are more general in the sense that some earlier published results are obtained from our result as special cases. Numerical results are given and illustrated graphically.

FORMULATION OF THE PROBLEM

The constitutive relation of an anisotropic and elastic solid is expressed by the generalized Hooke's law, which can be written as:

$\tau_{ij} = C_{ijkl}\, \varepsilon_{kl}$ i, j, k, l =1, 2, 3.

where, τ_{ij} are the Cartesian components of the stress and ε_{ij} is the strain tensor which is related with the displacement vector, u_i ; C_{ijkl} are the components of a fourth-order tensor called the elasticities of the medium. The Einstein convention for repeated indices is used.
In the absence of body forces, the field equations in the presence of voids may be taken as follows:

$$\tau_{ij,j} = \rho \ddot{u}_i, \tag{1}$$

$$\alpha\phi_{,ii} - \omega_0\phi - \varpi\dot{\phi} - \beta u_{i,i} = \rho\kappa\ddot{\phi} \tag{2}$$

$$\tau_{ij} = C_{ijkl}\varepsilon_{kl} + \beta\delta_{ij}\phi \tag{3}$$

In these equations, ϕ is the so-called volume fraction field. $\alpha, \beta, \omega_0, \varpi$ and κ are new material constants characterizing the presence of voids. ρ is the mass density. Comma followed by index shows partial derivative with respect to coordinate. The Einstein convention for repeated indices is used. Thus Above equation under gravity force G becomes:

$$C_{ijkl}u_{k,jl} + G_i + \beta\phi_{,i} = \rho\ddot{u}_i \qquad (4)$$

Medium is consisting of two homogeneous anisotropic fibre-reinforced semi-infinite elastic solid media M_1 and M_2 with different elastic and reinforcement parameters. The two media are perfectly welded in contact at a plane interface. Let us take orthogonal Cartesian axes $Ox_1x_2x_3$ with the origin at O. Ox_2 is pointing vertically upwards into the medium, M_1 ($x_2 > 0$). Each of the media M_1 ($x_2 > 0$) and M_2 ($x_2 < 0$) separated at $x_2 = 0$.

It is assumed that the waves travel in the positive direction of the x_1-axis and at any instant, all particles have equal displacements in any direction parallel to Ox_3. In view of those assumptions, the propagation of waves will be independent of x_3. Therefore all derivatives with respect to x_3 will be zero.

The general equation for a fibre-reinforced linearly elastic anisotropic media with respect to a direction $\bar{a} = (a_1, a_2, a_3)$ is as follows (Sengupta and Nath, 2001):

$$C_{ijkl}\varepsilon_{kl} = D_\lambda\varepsilon_{kk}\delta_{ij} + 2D_{\mu_T}\varepsilon_{ij} + D_\alpha(a_k a_m \varepsilon_{km}\delta_{ij} + \varepsilon_{kk}a_i a_{j)+} 2(D_{\mu_L} - D_{\mu_T})(a_i a_k \varepsilon_{kj} + a_j a_k \varepsilon_{ki}) + D_\beta(a_k a_m \varepsilon_{km} a_i a_j),$$

Strain tensor is $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ and D_λ, D_{μ_T} are elastic parameters. D_α, D_β and $(D_{\mu_L} - D_{\mu_T})$ are reinforced anisotropic viscoelastic parameters of higher order, s, defined as:

$$D_\lambda = \lambda_k\left(\frac{\partial}{\partial t}\right)^k \qquad D_\mu = \mu_k\left(\frac{\partial}{\partial t}\right)^k$$

$$D_\alpha = \alpha_k\left(\frac{\partial}{\partial t}\right)^k \qquad D_{\mu_L} = \mu_{L_k}\left(\frac{\partial}{\partial t}\right)^k$$

$$D_\beta = \beta_k\left(\frac{\partial}{\partial t}\right)^k \qquad D_{\mu_T} = \mu_{T_k}\left(\frac{\partial}{\partial t}\right)^k$$

$$k = 0, 1, 2 \ldots s.$$

An Einstein summation convention for repeated indices over "*k*" is used and comma followed by an index denotes the derivative with respect to coordinate.

u_i are the displacement vectors components. By choosing the fibre direction as $\bar{a} = (1, 0, 0)$, the components of stress becomes as follows:

$$\tau_{11} = (D_\lambda + 2D_\alpha + 4D_{\mu_L} - 2D_{\mu_T} + D_\beta)\varepsilon_{11} + (D_\lambda + D_\alpha)\varepsilon_{22} + (D_\lambda + D_\alpha)\varepsilon_{33} + \beta\phi,$$

$$\tau_{22} = (D_\lambda + D_\alpha)\varepsilon_{11} + (D_\lambda + 2D_{\mu_T})\varepsilon_{22} + D_\lambda\varepsilon_{33} + \beta\phi,$$

$$\tau_{33} = (D_\lambda + D_\alpha)\varepsilon_{11} + D_\lambda\varepsilon_{22} + (D_\lambda + 2D_{\mu_T})\varepsilon_{33} + \beta\phi,$$

$$\tau_{13} = 2D_{\mu_L}\varepsilon_{13},$$

$$\tau_{12} = 2D_{\mu_L}\varepsilon_{12},$$

$$\tau_{23} = 2D_{\mu_T}\varepsilon_{23}.$$

By choosing the fibre direction as $\bar{a} = (1, 0, 0)$; also by taking all derivatives w.r.t. x_3 zero. The Equation (4) of motion takes the following form:

$$(D_\lambda + 2D_\alpha + 4D_{\mu_L} - 2D_{\mu_T} + D_\beta)u_{1,11} + (D_\alpha + D_\lambda + D_{\mu_L})u_{2,21} + D_{\mu_L}u_{1,22} + \rho g u_{3,1} = \rho\ddot{u}_1 - \beta\phi_{,1} \qquad (5a)$$

$$(D_\alpha + D_{\lambda_k} + D_{\mu_L})u_{1,12} + D_{\mu_L}u_{2,11} + (D_{\lambda_k} + 2D_{\mu_T})u_{2,22} + \rho g u_{3,2} = \rho\ddot{u}_2 - \beta\phi_{,2} \qquad (5b)$$

$$D_{\mu_L}u_{3,11} + D_{\mu_T}u_{3,22} - \rho g(u_{1,1} + u_{2,2}) = \rho\ddot{u}_3, \qquad (5c)$$

From Equation (2), we have:

$$\alpha\left(\phi_{,11} + \phi_{,22}\right) - \omega_0\phi - \varpi\dot{\phi} - \beta\left(u_{1,1} + u_{2,2}\right) = \rho\kappa\ddot{\phi} \qquad (5d)$$

Similarly, we can get similar relations in M_2 with $\rho, \alpha, \beta, \kappa, c_v$ D_α, D_λ, D_{μ_L}, D_{μ_T} and D_β are replaced by $\rho', \alpha', \beta', \kappa', c'_v$ $D_{\alpha'}, D_{\lambda'}$, $D_{\mu'_L}$ $D_{\mu'_T}$ and $D_{\beta'}$, that is, all the parameters in medium M_1 are denoted by super script "dash".

Equations (5) in simplified form can be written as:

$$h_3u_{1,11} + h_2u_{2,21} + h_1u_{1,22} + \rho g u_{3,1} = \rho\ddot{u}_1 - \beta\phi_{,1} \qquad (6a)$$

$$h_4u_{2,22} + h_2u_{1,12} + h_1u_{2,11} + \rho g u_{3,2} = \rho\ddot{u}_2 - \beta\phi_{,2} \qquad (6b)$$

$$h_1 u_{3,11} + h_5 u_{3,22} - \rho g(u_{1,1} + u_{2,2}) = \rho \ddot{u}_3 \tag{6c}$$

$$\alpha\left(\phi_{,11} + \phi_{,22}\right) - \omega_0 \phi - \varpi \dot{\phi} - \beta\left(u_{1,1} + u_{2,2}\right) = \rho \kappa \ddot{\phi} \ , \tag{6d}$$

where

$$h_1 = D_{\mu_L}, \ ,$$
$$h_2 = \mathrm{D}_\alpha + D_\lambda + D_{\mu_L}, \ h_3 = \mathrm{D}_\lambda + 2D_\alpha + 4D_{\mu_L} - 2D_{\mu_T} + D_\beta$$
$$h_4 = D_{\lambda_k} + 2D_{\mu_T} \text{ and } h_5 = D_{\mu_T}$$

SOLUTION OF THE PROBLEM

To solve the coupled thermoelastic equations, we make the assumptions:

$$\begin{aligned} u_1, u_2, u_3 &= \hat{u}_1(x_2), \hat{u}_2(x_2), \hat{u}_3(x_2)\, exp\left\{i\omega(x_1 - ct)\right\} \\ \phi &= \ \hat{\phi}(x_2)\ exp\left\{i\omega(x_1 - ct)\right\} \end{aligned} \tag{7}$$

Thus coupled equations (6a, b and c)) becomes:

$$(\hbar_1 D^2 - \omega^2 \hbar_3 + \omega^2 \rho c^2)\hat{u}_1 + i\omega \hbar_2 D\hat{u}_2 + i\omega \rho g \hat{u}_3 + i\omega \beta \hat{\phi} = 0$$

$$(\hbar_5 D^2 - \hbar_1 \omega^2 + \rho \omega^2 c^2)\hat{u}_3 - \rho g(i\omega \hat{u}_1 + D\hat{u}_2) = 0$$

and

$$\left\{\alpha\left(D^2 - \omega^2\right) - \omega_0 + i\omega c \varpi + \omega^2 c^2 \rho \kappa\right\}\hat{\phi} - \beta\left(i\omega \hat{u}_1 + D\hat{u}_2\right) = 0$$

where

$$\hbar_1 = \mu_{Lk}(-i\,\omega c)^k, \ \hbar_2 = (\alpha_k + \lambda_k + \mu_{Lk})(-i\,\omega c)^k,$$
$$\hbar_3 = (\lambda_k + 2\alpha_k + 4\mu_{Lk} - 2\mu_{Tk} + \beta_k)(-i\,\omega c)^k,$$
$$\hbar_4 = (\lambda_k + 2\mu_{Tk})(-i\,\omega c)^k, \ \hbar_5 = \mu_{Tk}(-i\,\omega c)^k.$$

Above set of equation can be written as

$$\left.\begin{aligned} &(\hbar_1 D^2 - A_1)\hat{u}_1 + i\omega \hbar_2 D\hat{u}_2 + i\omega \rho g \hat{u}_3 + i\omega \beta \hat{\phi} = 0, \\ &(\hbar_4 D^2 - A_2)\hat{u}_2 + i\omega \hbar_2 D\hat{u}_1 + \rho g D\hat{u}_3 + \beta D\hat{\phi} = 0, \\ &\left(h_5 D^2 - A_2\right)\hat{u}_3 - \rho g(i\omega \hat{u}_1 + D\hat{u}_2) = 0 \\ &(D^2 - A_3)\hat{\phi} - \beta(i\omega \hat{u}_1 + D\hat{u}_2) = 0 \end{aligned}\right\} \tag{8}$$

where

$$A_1 = \omega^2\left(\hbar_3 - \rho c^2\right)$$
$$A_2 = \omega^2\left(\hbar_1 - \rho c^2\right)$$
$$A_3 = \omega^2 + \frac{\omega_0 - i\omega c \varpi - \omega^2 c^2 \rho \kappa}{\alpha}$$

From above set of equations, for non-trival solution, we have:

$$\begin{vmatrix} (\hbar_1 D^2 - A_1) & i\omega \hbar_2 D & i\omega \rho g & i\omega \beta \\ i\omega \hbar_2 D & (\hbar_4 D^2 - A_2) & \rho g D & \beta D \\ -i\omega \rho g & -\rho g D & (\hbar_5 D^2 - A_2) & 0 \\ -i\omega \beta & -\beta D & 0 & (D^2 - A_3) \end{vmatrix} (\hat{u}_1, \hat{u}_2, \hat{\phi}) = 0$$

This implies

$$(D^8 - ED^6 + FD^4 - GD^2 + H)(\hat{u}_1, \hat{u}_2, \hat{\phi}) = 0$$

where

$$E = \frac{1}{\hbar_1 \hbar_4 \hbar_5}\left\{\left(\hbar_1 A + \hbar_4 A_2 - \rho^2 g^2\right)\right\}$$
$$F = \frac{1}{\hbar_1 \hbar_4 \hbar_5}\left\{\rho^2 \omega^2 g^2\left(A_1 + 2\hbar_2 - \hbar_4\right) + \hbar_1 B + A_2 A\right\}$$
$$G = \frac{1}{\hbar_1 \hbar_4 \hbar_5}\left\{\rho^2 \omega^2 g^2\left(\left(A_1 + 2\hbar_2 - \hbar_4\right)A_3 - A_2\right) + A_2 B + \hbar_1 C\right\}$$
$$H = \frac{1}{\hbar_1 \hbar_4 \hbar_5}\left(\rho^2 \omega^2 g^2\left(A_2 A_3\right) + A_2 C\right)$$
$$A = \left(\hbar_4 A_1 + \hbar_1(A_2 + \hbar_4 A_3 - \beta^2) + \omega^2 \hbar_2^2\right)$$
$$B = \left\{\left(A_1 A_2 + \hbar_4 A_1 A_3 + \hbar_1 A_2 A_3 - \omega^2 \hbar_2^2 A_3\right) + \beta A_4(A_1 + 2\omega^2 \hbar_2 + \hbar_4 \omega^2)\right\}$$
$$C = \left(A_1 A_2 A_3 - \omega^2 A_2 \beta^2\right)$$
$$Let \ \ D^2 = m$$

Auxiliary equation becomes:

$$m^4 - Em^3 + Fm^2 - Gm + H = 0 \tag{9}$$

E,F,G and *H* must be positive for real positive roots (*m*). In the absence of gravity the above equation is cubic and if there are no voids then the above equation is quadratic in *m* and it is easy to solve.

Let mi (i=1,2,3,4) be four positive real roots, then solution by normal mode method has the following form:

$$\hat{u}_1 = \sum_{n=1}^{4} M_n \ e^{-m_n x_2}, \tag{10a}$$

$$\hat{u}_2 = \sum_{n=1}^{4} M_{1n} \ e^{-m_n x_2}, \tag{10b}$$

$$\hat{u}_3 = \sum_{n=1}^{4} M_{2n} \ e^{-m_n x_2}, \tag{10c}$$

$$\hat{\phi} = \sum_{n=1}^{4} M_{3n}\ e^{-m_n x_2}, \qquad (10d)$$

where M_n, M_{1n}, M_{2n} and M_{3n} are some parameters. By using Equations (10a to d) into Equations (8), we get the following relations:

$$M_{1n} = H_{1n} M_n\ ,$$
$$M_{2n} = H_{2n} M_n\ ,$$
$$M_{3n} = H_{3n} M_n\ ,$$

where

$$H_{1n} = \frac{i\,\omega(A_2 + \hbar_2 m_n^2 - \hbar_4 m_n^2)}{A_1 m_n - \hbar_2 \omega^2 m_n - \hbar_1 m_n^3},$$
$$H_{2n} = \frac{\hbar_1 m_n^2 - A_2}{\rho g\,(i\,\omega - m_n H_{1n})},$$
$$H_{3n} = \frac{A_3 - m_n^2}{\beta(i\,\omega - m_n H_{1n})}.$$

Hence we obtain the expressions of the displacement components, volume fraction field and stresses as follows

$$u_1 = \sum_{n=1}^{4} M_n\ e^{-m_n x_2}\ exp\{i\omega(x_1 - ct)\}, \qquad (11a)$$

$$u_2 = \sum_{n=1}^{4} H_{1n} M_n\ e^{-m_n x_2}\ exp\{i\omega(x_1 - ct)\}, \qquad (11b)$$

$$u_3 = \sum_{n=1}^{4} H_{2n} M_n\ e^{-m_n x_2}\ exp\{i\omega(x_1 - ct)\}, \qquad (11c)$$

$$\phi = \sum_{n=1}^{4} H_{3n} M_n\ e^{-m_n x_2}\ exp\{i\omega(x_1 - ct)\}, \qquad (11d)$$

Also it is found that

$$\tau_{12} = \sum_{n=1}^{4} \hbar_1 \left(-m_n + i\omega H_{1n}\right) M_n e^{-m_n x_2}\ exp\{i\omega(x_1 - ct)\}$$
$$\tau_{22} = \sum_{n=1}^{4} \{i\omega(\hbar_2 - \hbar_1) - \hbar_4 m_n H_{1n} - \beta H_{3n}\} M_n e^{-m_n x_2}\ exp\{i\omega(x_1 - ct)\}$$
$$\tau_{23} = \sum_{n=1}^{4} \hbar_5 \left(-m_n H_{2n}\right) M_n e^{-m_n x_2}\ exp\{i\omega(x_1 - ct)\}, \qquad where\ \hbar_5 = \mu_{Tk}(-i\omega c)^k$$

Similar expressions can be obtained for second mediun and present them with super script dashes as follows:

$$u_1' = \sum_{n=1}^{4} M_n'\ e^{-m_n' x_2}\ exp\{i\omega(x_1 - ct)\}, \qquad (12a)$$

$$u_2' = \sum_{n=1}^{4} H_{1n}' M_n'\ e^{-m_n' x_2}\ exp\{i\omega(x_1 - ct)\}, \qquad (12b)$$

$$u_3' = \sum_{n=1}^{4} H_{2n}' M_n'\ e^{-m_n' x_2}\ exp\{i\omega(x_1 - ct)\}, \qquad (12c)$$

$$\phi' = \sum_{n=1}^{4} H_{3n}' M_n'\ e^{-m_n' x_2}\ exp\{i\,\omega(x_1 - ct)\}. \qquad (12d)$$

Also it is found that:

$$\tau_{12}' = \sum_{n=1}^{4} \hbar_1' \left(-m_n' + i\,\omega H_{1n}'\right) M_n' e^{-m_n' x_2}\ exp\{i\,\omega(x_1 - ct)\},$$
$$\tau_{22}' = \sum_{n=1}^{4} \{i\,\omega(\hbar_2' - \hbar_1') - \hbar_4 m_n' H_{1n}' - \beta' H_{3n}'\} M_n' e^{-m_n' x_2}\ exp\{i\,\omega(x_1 - ct)\},$$
$$\tau_{23}' = \sum_{n=1}^{4} \hbar_5 \left(-m_n' H_{2n}'\right) M_n' e^{-m_n' x_2}\ exp\{i\,\omega(x_1 - ct)\}.$$

In order to determine the secular equations, we have the following boundary conditions.

BOUNDARY CONDITIONS

1. The displacement components and volume fraction field between the mediums are continuous, that is, $u_1 = u_1', u_2 = u_2'$, $u_3 = u_3'$ and $\phi = \phi'$ on $x_2 = 0$, for all x_1 and *t.*
2. Stress continuity exists, i.e. $\tau_{12} = \tau_{12}'$, $\tau_{22} = \tau_{22}'$, $\tau_{23} = \tau_{23}'$ on $x_2 = 0$, for all x_1 and *t.*
3. It is assumed that the following relation hold:

$$\left(\frac{\partial \phi}{\partial x_2} + h\phi\right)_{medium\,M_1} = \left(\frac{\partial \phi'}{\partial x_2} + h\phi'\right)_{medium\ M_2},\ \text{on the plane}\ x_2 = 0\ ,\forall\, x_1\ and\ t,$$

where h is a constant.
Boundary conditions implies the following equations:

$$\left.\begin{array}{l} M_1 + M_2 + M_3 + M_4 = M_1' + M_2' + M_3' + M_4' \\ H_{11}M_1 + H_{12}M_2 + H_{13}M_3 + H_{14}M_4 = H_{11}'M_1' + H_{12}'M_2' + H_{13}'M_3' + H_{14}'M_4' \\ H_{21}M_1 + H_{22}M_2 + H_{23}M_3 + H_{24}M_4 = H_{21}'M_1' + H_{22}'M_2' + H_{23}'M_3' + H_{24}'M_4' \\ H_{31}M_1 + H_{32}M_2 + H_{33}M_3 + H_{34}M_4 = H_{31}'M_1' + H_{32}'M_2' + H_{33}'M_3' + H_{34}'M_4' \end{array}\right\} \qquad (13a)$$

$$\left.\begin{aligned}
&\sum_{n=1}^{4} \hbar_1\left(-m_n+i\omega H_{1n}\right)M_n=\sum_{n=1}^{4} \hbar_1'\left(-m_n'+i\omega H_{1n}'\right)M_n' ,\\
&\sum_{n=1}^{4}\left\{i\omega(\hbar_2-\hbar_1)-\hbar_4 m_n H_{1n}-\beta H_{2n}\right\}M_n=\\
&\qquad\sum_{n=1}^{4}\left\{i\omega(\hbar_2'-\hbar_1')-\hbar_4' m_n' H_{1n}'-\beta' H_{2n}'\right\}M_n' ,\\
&\sum_{n=1}^{4}\hbar_5\left(-i\omega c\right)^k\left(-m_n H_{2n}\right)M_n=\sum_{n=1}^{4}\hbar_5'\left(-m_n' H_{2n}'\right)M_n'\\
&\sum_{n=1}^{4}(h-m_n)H_{3n}M_n=\sum_{n=1}^{4}(h'-m_n')H_{3n}'M_n'
\end{aligned}\right\} \quad (13b)$$

Elimination of constants $M_n \; and \; M_n', \; (n=1,2,3,4)$ from above set of relation, gives the following secular equation for surface wave in a fibre reinforced viscoelastic material of higher order *s* under gravity with voids.

$$\det(a_{pq})=0; \quad p=q=1,2,3,4,5,6,7,8. \tag{14}$$

where

$$a_{11}=1,\; a_{12}=1,\; a_{13}=1, a_{14}=1,\; a_{15}=-1,\; a_{16}=-1,\; a_{17}=-1,\; a_{18}=-1$$
$$a_{21}=H_{11},\; a_{22}=H_{12},\; a_{23}=H_{13},\; a_{24}=H_{14},\; a_{25}=-H_{11}',\; a_{26}=-H_{12}',\; a_{27}=-H_{13}',\; a_{28}=-H_{14}',$$
$$a_{31}=H_{21},\; a_{32}=H_{22},\; a_{33}=H_{23},\; a_{34}=H_{24},\; a_{35}=-H_{21}',\; a_{36}=-H_{22}',\; a_{37}=-H_{23}',\; a_{38}=-H_{24}',$$
$$a_{41}=H_{31},\; a_{42}=H_{32},\; a_{43}=H_{33},\; a_{44}=H_{34},\; a_{45}=-H_{31}',\; a_{46}=-H_{32}',\; a_{47}=-H_{33}',\; a_{48}=-H_{34}',$$

$$a_{51}=\hbar_1\left(-m_1+i\omega H_{11}\right),\qquad a_{52}=\hbar_1\left(-m_2+i\omega H_{12}\right),$$
$$a_{53}=\hbar_1\left(-m_3+i\omega H_{13}\right),\qquad a_{54}=\hbar_1\left(-m_3+i\omega H_{14}\right),$$
$$a_{55}=-\hbar_1'\left(-m_1'+i\omega H_{11}'\right),\qquad a_{56}=-\hbar'\left(-m_2'+i\omega H_{12}'\right),$$
$$a_{57}=-\hbar'\left(-m_3'+i\omega H_{13}'\right),\qquad a_{58}=-\hbar'\left(-m_4'+i\omega H_{14}'\right)$$
$$a_{61}=\left\{i\omega(\hbar_2-\hbar_1)-\hbar_4 m_1 H_{11}-\beta H_{21}\right\},$$
$$a_{62}=\left\{i\omega(\hbar_2-\hbar_1)-\hbar_4 m_2 H_{12}-\beta H_{22}\right\},$$
$$a_{63}=\left\{i\omega(\hbar_2-\hbar_1)-\hbar_4 m_3 H_{13}-\beta H_{23}\right\},$$
$$a_{64}=\left\{i\omega(\hbar_2-\hbar_1)-\hbar_4 m_3 H_{14}-\beta H_{24}\right\}$$
$$a_{65}=-\left\{i\omega(\hbar_2'-\hbar_1')-\hbar_4' m_1' H_{11}'-\beta' H_{21}'\right\},$$
$$a_{66}=-\left\{i\omega(\hbar_2'-\hbar_1')-\hbar_4' m_2' H_{12}'-\beta' H_{22}'\right\},$$
$$a_{67}=-\left\{i\omega(\hbar_2'-\hbar_1')-\hbar_4' m_3' H_{13}'-\beta' H_{23}'\right\},$$
$$a_{68}=-\left\{i\omega(\hbar_2'-\hbar_1')-\hbar_4' m_4' H_{14}'-\beta' H_{24}'\right\}$$
$$a_{71}=\left\{\hbar_5 m_1 H_{21}\right\},\; a_{72}=\left\{\hbar_5 m_2 H_{22}\right\},$$
$$a_{73}=\left\{\hbar_5 m_3 H_{23}\right\},\; a_{74}=\left\{\hbar_5 m_4 H_{24}\right\},$$
$$a_{75}=-\left\{\hbar_5' m_1' H_{21}'\right\},\; a_{76}=-\left\{\hbar_5' m_2' H_{22}'\right\},$$
$$a_{77}=-\left\{\hbar_5' m_3' H_{23}'\right\},\; a_{78}=-\left\{\hbar_5' m_4' H_{24}'\right\},$$
$$a_{81}=(h-m_1)H_{31},\quad a_{82}=(h-m_2)H_{32},\quad a_{83}=(h-m_3)H_{33},\quad a_{84}=(h-m_4)H_{34},$$
$$a_{85}=-(h'-m_1')H_{31}',\; a_{86}=-(h'-m_2')H_{32}',\; a_{87}=-(h'-m_3')H_{33}',\; a_{88}=-(h'-m_4')H_{34}'.$$

PARTICULAR CASES

Stoneley waves

Equation (14) is the secular equation for Stonely waves in a fibre reinforced viscoelastic media of higher order. For k = 0, results are similar to Abd-Alla (2003). If rotational, voids and fiber-reinforced parameters are ignored, then for k = 0, the results are same as Stoneley (1924).

Rayleigh waves

Rayleigh wave is a special case of the above general surface wave. In this case we consider a model where the medium, M_1 is replaced by vacuum. Since the boundary, $x_2=0$ is adjacent to vacuum. It is free from surface traction. So the stress boundary condition in this case may be expressed as:

$\tau_{12}=0$, $\tau_{22}=0$ on $x_2=0$, for all x_1 and *t*.

$$\frac{\partial\phi}{\partial x_2}+h\phi=0,\ \text{on the plane}\ x_2=0\ ,\forall\, x_1\ and\ t,$$

It is assumed that gravitational field produces a hydrostatic initial stress. It produced by a slow process of creep where the shearing stresses tend to small or vanish after a long period of time. Equilibroim conditions of initial stress are:

$$\frac{\partial\tau_{11}}{\partial x_1}=0,\ \frac{\partial\tau_{11}}{\partial x_2}+\rho g=0$$

Thus above set of equations reduces to:

$$\sum_{n=1}^{4} \hbar_1 \left(-m_n + i\omega H_{1n}\right) M_n = 0 \,,$$

$$\sum_{n=1}^{4} \left\{ i\omega(\hbar_2 - \hbar_1) - \hbar_4 \left(-i\omega c\right)^k m_n H_{1n} - \beta H_{2n} \right\} M_n = 0,$$

$$\sum_{n=1}^{4} (m_n - h) H_{3n} M_n = 0,$$

$$\sum_{n=1}^{4} \left\{ i\omega\hbar_3 - (\hbar_2 - \hbar_1) m_n H_{1n}) - \beta H_{3n} \right\} M_n = 0$$

Eliminating the constants M_1, M_2, M_3 and M_4 we get the wave velocity equation for Rayleigh waves in the fibre-reinforced viscoelastic media of order *s* under the influence of gravity as follows:

$$\det(b_{lm}) = 0; \quad l = m = 1, 2, 3, 4. \tag{15}$$

where

$$b_{11} = \hbar_1 \left(-m_1 + i\omega H_{11}\right), \quad b_{12} = \hbar_1 \left(-m_2 + i\omega H_{12}\right),$$

$$b_{13} = \hbar_1 \left(-m_3 + i\omega H_{13}\right), \quad b_{14} = \hbar_1 \left(-m_3 + i\omega H_{14}\right),$$

$$b_{21} = \left\{ i\omega(\hbar_2 - \hbar_1) - \hbar_4 m_1 H_{11} - \beta H_{21} \right\},$$

$$b_{22} = \left\{ i\omega(\hbar_2 - \hbar_1) - \hbar_4 m_2 H_{12} - \beta H_{22} \right\},$$

$$b_{23} = \left\{ i\omega(\hbar_2 - \hbar_1) - \hbar_4 m_3 H_{13} - \beta H_{23} \right\},$$

$$b_{24} = \left\{ i\omega(\hbar_2 - \hbar_1) - \hbar_4 \left(-i\omega c\right)^k m_3 H_{14} - \beta H_{24} \right\}$$

$$b_{31} = (m_1 - h) H_{31}, \quad b_{32} = (m_2 - h) H_{32},$$

$$b_{33} = (m_3 - h) H_{33}, \; b_{34} = (m_4 - h) H_{34},$$

$$b_{41} = \left\{ i\omega\hbar_3 - (\hbar_2 - \hbar_1) m_1 H_{11}) - \beta H_{31} \right\},$$

$$b_{42} = \left\{ i\omega\hbar_3 - (\hbar_2 - \hbar_1) m_2 H_{12}) - \beta H_{32} \right\},$$

$$b_{43} = \left\{ i\omega\hbar_3 - (\hbar_2 - \hbar_1) m_3 H_{13}) - \beta H_{33} \right\},$$

$$b_{44} = \left\{ i\omega\hbar_3 - (\hbar_2 - \hbar_1) m_4 H_{14}) - \beta H_{34} \right\}.$$

Equation (15) is the secular equation for Rayleigh wave for the medium M_1. For k = 0 and by ignoring the voids and gravitational effects our results are same as that of Sengupta and Nath (2001). If one ignores the fibre-reinforced parameters also then results are same as Rayleigh (1885).

NUMERICAL SIMULATION AND DISCUSSION

The following values of elastic constants are considered Chattopadhyay et al. (1987) for mediums M and M_1 respectively.

$\rho = 2660 Kg/m^3$, $\lambda = 5.65\times10^{10}\, Nm^{-2}$, $\mu_T = 2.46\times10^{9}\, Nm^{-2}$, $\mu_L = 5.66\times10^{9}\, Nm^{-2}$,
$\alpha = -1.28\times10^{9}\, Nm^{-2}$, $\beta = 220.90\times10^{9}\, Nm^{-2,}$

$\rho = 7800 Kg/m^3$, $\lambda = 5.65\times10^{9}\, Nm^{-2}$, $\mu_T = 2.46\times10^{10}\, Nm^{-2}$, $\mu_L = 5.66\times10^{10}\, Nm^{-2}$,
$\alpha = -1.28\times10^{10}\, Nm^{-2}$, $\beta = 220.90\times10^{10}\, Nm^{-2,}$

The numerical technique outlined above was used to obtain secular equation, surface waves velocity and attenuation coefficients under the effects of rotation in two models with voids.

For the sake of brevity some computational results are being presented here. The variations are shown in Figures 1 and 2, respectively.

Figure 1a to i show the variation of the magnitude of the frequency equation $|\Delta|$, Stoneley wave velocity $\mathrm{Re}(|\Delta|)$ and attenuation coefficient $\mathrm{Im}(|\Delta|)$ with respect to the frequency ω for different values of order k, gravity field g and phase velocity c. The magnitude of the frequency equation increases with increasing of frequency, while it decreases with increasing of order and gravity field and when effect of phase velocity it increases with increasing of phase velocity, as well, Stoneley wave velocity decreases with increasing of frequency, while it increases with increasing of order and gravity field and when effect of phase velocity, it decreases with increasing of phase velocity and the attenuation coefficient increases with increasing of frequency, except when effect of phase velocity it decreases with increasing of frequency, while it increases with increasing of order , as well it decreases with increasing of gravity field and phase velocity.

Figures 2a to i show the variation of the magnitude of the frequency equation $|\Delta|$, Stoneley wave velocity $\mathrm{Re}(|\Delta|)$ and attenuation coefficient $\mathrm{Im}(|\Delta|)$ with respect to the frequency ω for different values of order k, gravity field g and phase velocity c. The magnitude of the frequency equation increases with increasing of frequency, while it decreases with increasing of order and gravity field and when effect of phase velocity it increases with increasing of phase velocity, as well, Stoneley wave velocity decreases with increasing of frequency, while it increases with increasing of order and gravity field and when effect of phase velocity, it decreases with increasing of phase velocity and the attenuation coefficient increases with increasing of frequency and when effect of phase velocity it increases and decreases gradually with increasing of frequency, while it decreases with increasing of phase velocity.

Finally, one can see that there is a similarity between

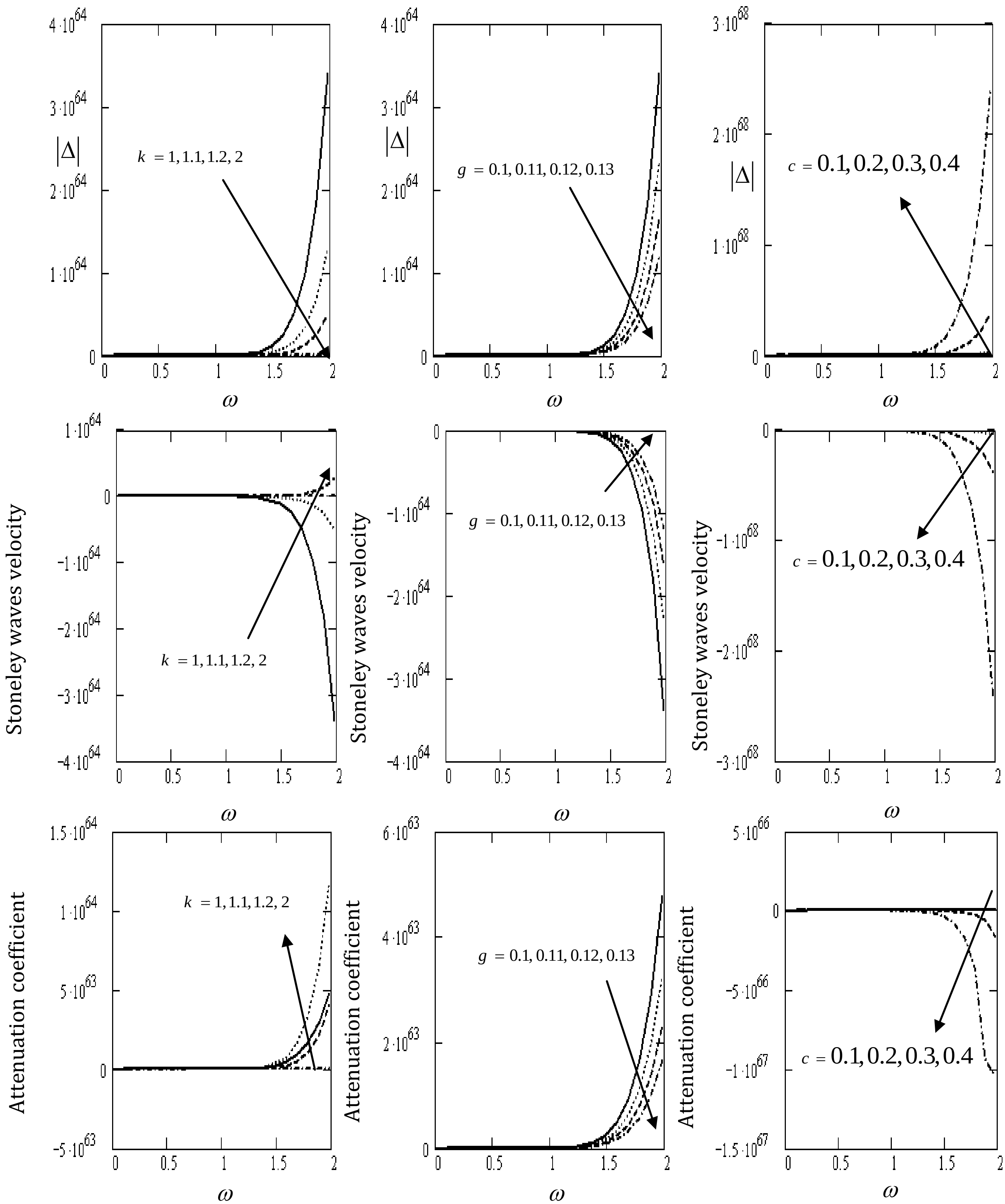

Figure 1. Variation of $|\Delta|$, velocity ($\mathrm{Re}(|\Delta|)$) and attenuation coefficient ($\mathrm{Im}(|\Delta|)$) for Stoneley waves with respect to ω with variation of k, g and c.

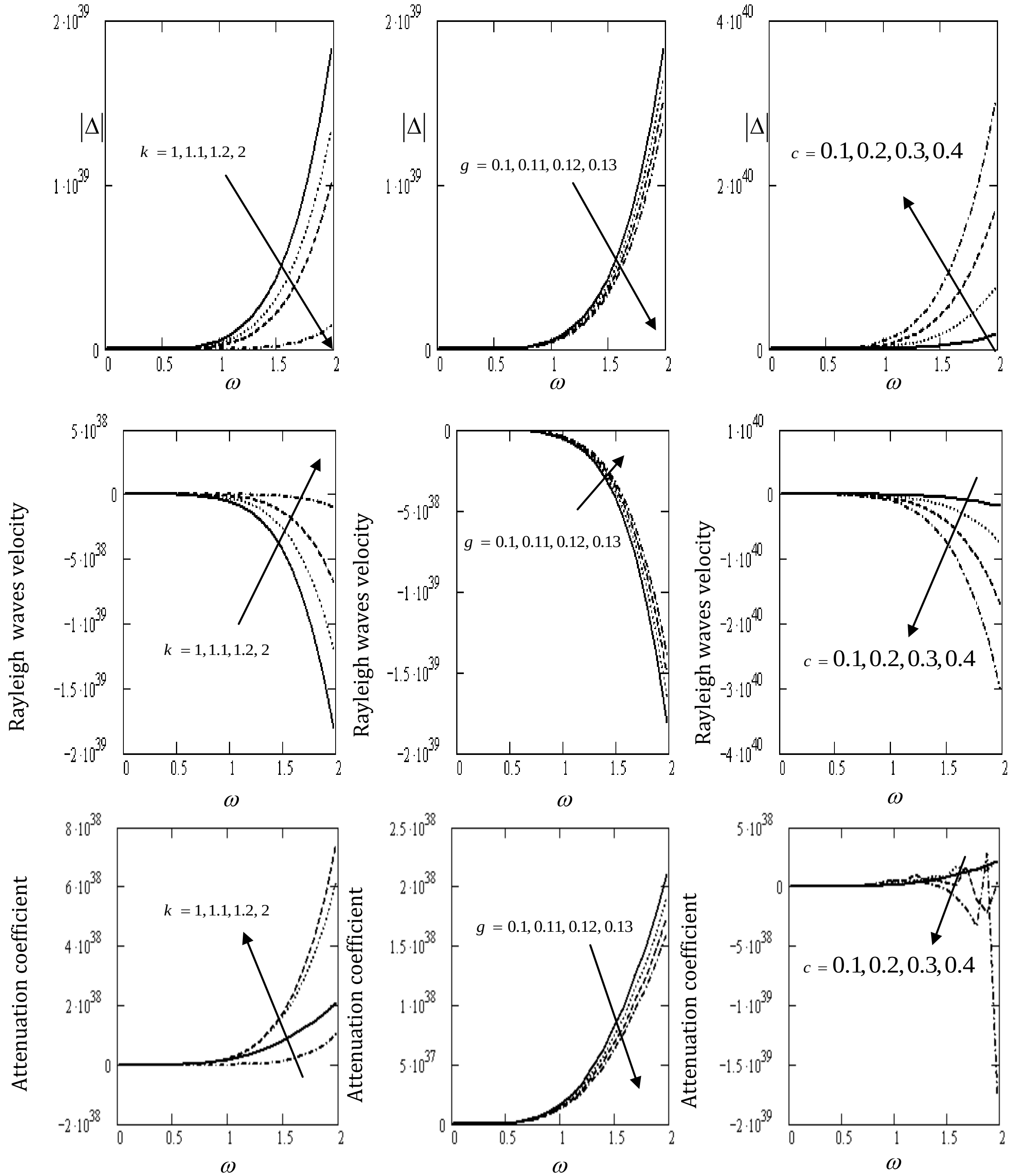

Figure 2. Variation of $|\Delta|$, velocity ($\mathrm{Re}(|\Delta|)$) and attenuation coefficient ($\mathrm{Im}(|\Delta|)$) for Rayleigh waves with respect to ω with variation of k, g and c.

the graphs of two waves types (that is, Stoneley and Rayleigh) in the behavior but there are differences between the values and part of their behavior.

CONCLUSION

Due to the complicated nature of the governing equations of the fibre-reinforced anisotropic general viscoelastic media of higher order with voids, the work done in this field is unfortunately limited in number. The method used in this study provides a quite successful in dealing with such problems. This method gives exact solutions in the fibre-reinforced anisotropic elastic media without any assumed restrictions on the actual physical quantities that appear in the governing equations of the problem considered. Important phenomena are observed in all these computations:

1. It was found that the solutions obtained in the context of the fibre-reinforced anisotropic general viscoelastic media of higher integer and fractional order with voids, however, exhibit the behavior of speeds of wave propagation.
2. By comparing Figures 1 and 2, it is found that the wave velocity has the same behavior in both media. But with the passage of gravity field, numerical values of wave velocity in the viscoelastic media are large in comparison due to the viscoelastic fiber-reinforced.
3. Special cases are considered as Stoneley and Rayleigh waves only.
4. The results presented in this paper should prove useful for researchers in material science, designers of new materials.
5. Study of the phenomenon of rgravity field is also used to improve the conditions of oil extractions. Finally, if the rotation is neglected, the relevant results obtained are deduced to the results obtained by Sengupta and Nath (2001).

REFERENCES

Abd-Alla AM (1999). Propagation of Rayleigh waves in an elastic half-space of orthotropic material. Appl. Math. Comput. 99:61-69.

Abd-Alla AM, Abo-Dahab SM, Al-Thamali TA (2012). Propagation of Rayleigh waves in a rotating orthotropic material elastic half-space under initial stress and gravity. J. Mech. Sci. Technol. 26:2815-2823.

Abd-Alla AM, Abo-Dahab SM, Bayones FS (2013). Propagation of Rayleigh waves in magneto-thermo-elastic half-space of a homogeneous orthotropic material under the effect of rotation, initial stress and gravity field. J. Vib. Control 19(9):1395–1420.

Abd-Alla AM, Abo-Dahab SM, Hammad HAH (2011). Propagation of Rayleigh waves in generalized magneto-thermoelastic orthotropic material under initial stress and gravity field. Appl. Math. Modelling. 35:2981-3000.

Abd-Alla AM, Ahmed SM (1996). Rayleigh waves in an orthotropic thermoelastic medium under gravity field and initial stress. Earth, Moon, Planets 75:185-197.

Abd-Alla AM, Ahmed SM (1999). Propagation of love waves in a non-homogeneous orthotropic elastic layer under initial stress overlying semi-infinite medium. Appl. Math. Comput. 106:265-275.

Abd-Alla AM, Ahmed SM (2003). Stoneley and Rayleigh waves in a non-homogeneous orthotropic elastic medium under the influence of gravity. Appl. Math. Comput. 135:187-200.

Abd-Alla AM, Hammad HAH, Abo-Dahab SM (2004). Rayleigh waves in a magnetoelastic half-space of orthotropic material under influence of initial stress and gravity field. Appl. Math. Comput. 154:583-597.

Abd-Alla AM, Nofal TA, Abo-Dahab SM, Al-Mullise A (2013). Surface waves propagation in fibre-reinforced anisotropic elastic media subjected to gravity field. Int. J. Phys. Sci. 8(14):574-584.

Abo-Dahab SM (2010). Propagation of P waves from stress-free surface elastic half-space with voids under thermal relaxation and magnetic field. Appl. Math. Model. 34:1798-1806.

Abo-Dahab SM, Abd-Alla AM (2014). Effects of voids and rotation on plane waves in generalized thermoelasticity. J. Mech. Sci. Technol. 27:3607-3614.

Belfield AJ, Rogers TG, Spencer AJM (1983). Stress in elastic plates reinforced by fibre lying in concentric circles. J. Mech. Phys. Solids. 31:25-54.

Bromwich J (1898). On the influence of gravity on elastic waves, and, in particular, on the vibrations of an elastic globe. Proc. London Math. Soc. 30:98-120.

Bullen KE (1965). An introduction to the theory of seismology. London: Cambridge University press. pp. 85-99.

Chandrasekharaiah DS (1987a). Effects of surface stresses and voids on Rayleigh waves in an elastic solid. Int. J. Eng. Sci. 25(2):205-211.

Chandrasekharaiah DS (1987b). Plane waves in a rotating elastic solid with voids. Int. J. Engng. Sci. 25(5):591-596.

Cowin SC, Nunziato JW (1983). Linear elastic materials with voids. J. Elasticity 13(2):125-147.

De SK, Sengupta PR (1974). Influence of gravity on wave propagation in an elastic layer. J. Acanst. Soc. Am. 55:5-13.

Elnaggar AM, Abd-Alla AM (1989). Rayleigh waves in magneto-thermo-microelastic half-space under initial stress. Earth, Moon, and Planets, 45(2):175-185.

Ewing WM, Jardetzky WS (1957). Elastic waves in layered media. (New York, Toronto, London: McGraw Hill) Press F. pp. 348- 350.

Fonseca I, Fusco N, Leoni G, Millot V (2011). Material voids in elastic solids with anisotropic surface energies. J. de Mathé. Pures Appliquées 96:591-639.

Iesan D (1986). A Theory of thermoelastic materials with voids. Acta Mechanica. 60(1-2):67- 89.

Kakar R, Kakar S, Kaur K (2013). Rayleigh, Love and Stoneley waves in fiber-reinforced,general viscoelastic media of higher order under gravity. Int. J. Phys. Math. Sci. 4(1):53-61.

Nunziato JW, Cowin SC (1979). A nonlinear theory of elastic materials with voids. Arch. Rational Mech. Anal. 72(2):175-201.

Pal KC, Sengupta PR (1987). Surface waves in visco-elastic media of general type in the presence of thermal field and gravity. Proc. Indian Nat. Sci. Acad. A53:353-372.

Puri P, Cowin SC (1985). Plane waves in linear elastic materials with voids. J. Elast. 15(2):167-183.

Rayleigh L (1885). On wave propagation along the plane surface of an elastic solid. Proc. London. Math. Soc. 17:4-11.

Ren J, Li H, Cheng C, Yuan X (2012). Coupling effects of void shape and void size on the growth of an elliptic void in a fiber-reinforced hyper-elastic thin plate. Acta Mechanica Solida Sinica. 25:312-320.

Sarvajit S (2002). Comments on “Surface waves in fibre-reinforced anisotropic elastic media”. by Sengupta and Nath, Sãdhanã 27(3):405-407.

Sengupta PR, Acharya D (1979). the influence of gravity on the propagation of waves in a thermoelastic layer. Rev. Romm. Sci. Technol. Mech. Appl. Tome. 24:395-406.

Sengupta PR, Nath S (2001). Surface waves in fibre-reinforced anisotropic elastic media. Sãdhanã 26:363-370.

Singh B, Pal R (2011). Surface wave propagation in a generalized thermoelastic material with voids. Appl. Math. 2:521-526.

Stoneley R (1924). The elastic waves at the surface of separation of two solids. Proc. R. Soc. London. A106:416-420.

Tvergaard V (2011). Elastic–plastic void expansion in near-self-similar shapes. Comput. Mater. Sci. 50:3105-3109.

Vishwakarma SK, Gupta S, Majhi DK (2013). Influence of rigid boundary on the love wave propagation in elastic layer with void pores. Acta Mechanica Solida Sinica. 26:551-558.

International Journal of Physical Sciences

Related Journals Published by Academic Journals

- ***African Journal of Pure and Applied Chemistry***
- ***Journal of Internet and Information Systems***
- ***Journal of Geology and Mining Research***
- ***Journal of Oceanography and Marine Science***
- ***Journal of Environmental Chemistry and Ecotoxicology***
- ***Journal of Petroleum Technology and Alternative Fuels***

academicJournals